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PROPERTIES AND TABLES OF THE EXTENDED
AIRY-HARDY INTEGRALS

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RESEARCH LABORATORY OF ELECTRONICS
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Abstract and Foreword

1. Airy-Hardy and other associated functions play an important role in the theory of approximate integration; specifically, in saddlepoint methods. They appear in the pure and mixed transitional cases corresponding to configurations that lead to saddlepoints of third order. (For definitions and other details see references 1 and 2.)

There are three basic Airy-Hardy functions. They are denoted by: $Ah_{1,3}$, $Ah_{2,3}$, $Ah_{3,3}$. The first index refers to the particular function; the second index expresses the order of the saddlepoint associated with the function. Definitions, series representations, and properties are given in this report. The connections of these functions with the methods of integration are supplied in Appendices I and II.

2. The functions $Ah_{\nu,3}$, $\nu = 1, 2, 3$, correspond to the so-called pure transitions. The function $Ah_{1,3}(s)$ has been tabulated for complex values of the argument s . This function represents a surface over the s -plane. Radial cross sections have been computed for angular arguments in steps from 0° to 180° and for $|s|$ in the range from zero to 4. Twenty-seven tables give $|Ah_{1,3}(s)|$, $\arg \{Ah_{1,3}(s)\}$, $\operatorname{Re}\{Ah_{1,3}(s)\}$, and $\operatorname{Im}\{Ah_{1,3}(s)\}$. The plot of $Ah_{1,3}(s)$ is expressed in the graphs of the cross sections of each of the functions above. Several isometric plots of the corresponding surfaces are also given.

3. The aim of the computations given in this report is to show an over-all behavior of the functions indicated in paragraph 2 above, rather than to produce a detailed evaluation of those functions. The computations shown in these tables were made on small mechanical machines during the summer of 1950.

4. Only the function $Ah_{1,3}$ was tabulated. There are simple connecting relations between this function and $Ah_{2,3}$ and $Ah_{3,3}$ that permit a rapid computation for $Ah_{2,3}$ and $Ah_{3,3}$ from the tables of $Ah_{1,3}$.

5. Mixed transitions of third order require some other functions, $\phi_{\nu,p}(B)$ and $\phi_{\nu,o}(B)$, which appear when the saddlepoint of third order runs over or in the vicinity of a pole or a zero of the integrand. (See Appendices I and II.) These functions can be expressed as the integrals or derivatives, respectively, of the $Ah_{1,3}$ functions. The functions, $\phi_{\nu,p}(B)$ and $\phi_{\nu,o}(B)$, are not tabulated in this report; however, the corresponding series representations are given. A simple numerical integration or differentiation can be performed from $Ah_{1,3}$, and the evaluation of $\phi_{1,p}(B)$ and $\phi_{1,o}(B)$ obtained without much effort. The figures of transient composition given in Appendix II were obtained with ease by this method.

6. This report was originally planned as a part of the Series 55 reports. Later, its publication as a separate report seemed appropriate because the material, which is self-contained, is applicable in other fields and may prove more useful if it is published separately.

7. The authors want to express their thanks to Mrs. Ann Moldauer for her help in preparing the numerical computations involved in this report; to Miss Elizabeth Campbell for her constant advice and suggestions; and to other members of the Joint Computing Group for their careful proofreading of the tables.

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Manuel V. Cerrillo
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I. PROPERTIES OF EXTENDED AIRY-HARDY INTEGRALS

I.1 DEFINITIONS OF THE FUNCTIONS AND THEIR INTERRELATIONS

In certain branches of applied mathematics – in a new theory of the transient analysis of linear electrical systems⁽¹⁾, in particular – there arise a number of important complex prototype integrals, the evaluation of which can be accomplished only in terms of little-known functions. One set of these integrals, the Lommel generating functions, was mentioned in a previous report⁽²⁾, and discussed and tabulated in greater detail in another report⁽³⁾.

A second set of integrals of basic importance is:

$$Ah_{\nu,3}(B) = \frac{1}{2\pi i} \int_{\gamma_\nu} e^{Bz+z^3} dz \quad ; \quad \nu = 1, 2, \text{ and } 3 \quad 1(I.1)$$

The contours γ_1 , γ_2 , and γ_3 corresponding to the three functions $Ah_{1,3}(B)$, $Ah_{2,3}(B)$, and $Ah_{3,3}(B)$ are those shown in Fig. 1a, b, and c. It is apparent that

$$Ah_{1,3}(B) = Ah_{2,3}(B) + Ah_{3,3}(B) \quad 2(I.1)$$

Since the integrand is an entire function, these contours may be distorted from the positions shown as long as they pass to infinity in directions that do not deviate more than 30° from the directions of Fig. 1. Under these conditions, the integrals converge for all finite B , real or complex, and therefore define three single-valued, entire functions.

To establish other relations between the three functions, let $z \rightarrow ze^{\pm i2\pi/3}$ in the integrals for $Ah_{2,3}(B)$ and $Ah_{3,3}(B)$. Then

$$\left. \begin{array}{l} Ah_{2,3}(B) \\ Ah_{3,3}(B) \end{array} \right\} = \frac{1}{2\pi i} \int_{-\gamma_1} e^{Be^{\pm i2\pi/3} z + z^3} dz \cdot e^{\pm i2\pi/3}$$

or

$$\left. \begin{array}{l} Ah_{2,3}(B) \\ Ah_{3,3}(B) \end{array} \right\} = -e^{\pm i2\pi/3} \cdot Ah_{1,3}(Be^{\pm i2\pi/3}) \quad 3(I.1)$$

The three functions are, therefore, not independent of one another. The knowledge of one determines the other two completely.

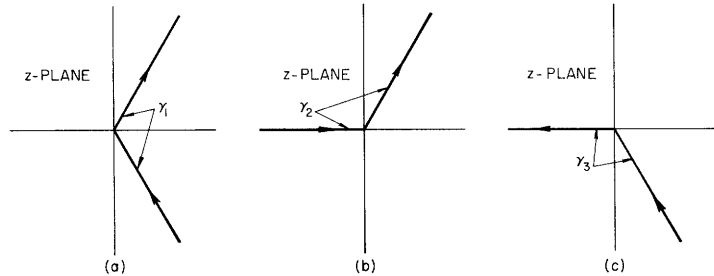


Fig. 1. Contours pertinent to the evaluation of extended Airy-Hardy integrals.

I. 2 POWER SERIES EXPANSION

For purposes of computation, it is helpful to have the power series expansion of, let us say, $Ah_{1,3}(B)$. The uniform convergence of the development

$$e^{Bz} = \sum_{k=0}^{\infty} \frac{(Bz)^k}{k!}$$

guarantees that if this series is substituted in the first integral of 1(I. 1) the processes of integration and summation can be interchanged. The resultant sum

$$Ah_{1,3}(B) = \sum_{k=0}^{\infty} \frac{B^k}{k!} \frac{1}{2\pi i} \int_{\gamma_1} z^k e^{z^3} dz$$

can be evaluated term by term. If we let $z = re^{-i\pi/3}$ on the lower part of the contour (Fig. 1a), and $z = re^{i\pi/3}$ on the upper part, we obtain

$$Ah_{1,3}(B) = \sum_{k=0}^{\infty} \frac{B^k}{k!} \frac{1}{2\pi i} \left\{ e^{-i(k+1)\pi/3} \int_{\infty}^0 r^k e^{-r^3} dr + e^{i(k+1)\pi/3} \int_0^{\infty} r^k e^{-r^3} dr \right\}$$

Hence,

$$Ah_{1,3}(B) = \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{B^k}{k!} \sin \left[(k+1) \frac{\pi}{3} \right] \int_0^{\infty} r^k e^{-r^3} dr$$

$$Ah_{1,3}(B) = \frac{\sqrt{3}}{6\pi} \sum_{k=0}^{\infty} \left\{ \frac{2}{\sqrt{3}} \sin \left[(k+1) \frac{\pi}{3} \right] \right\} \frac{\Gamma\left(\frac{k+1}{3}\right)}{k!} B^k \quad 1(I. 2)$$

This is the desired power series. The quantity in braces takes on the values 1, 1, 0, -1, -1, 0, ... cyclically as $k = 0, 1, 2, \dots$. At $B = 0$,

$$Ah_{1,3}(0) = \frac{\sqrt{3}}{6\pi} \Gamma(1/3) = 0.246162704$$

The radius of convergence of 1(I. 2) is

$$R = \lim_{k \rightarrow \infty} \left| \Gamma\left(\frac{k+1}{3}\right) / k! \right|^{-1/k} = \infty$$

as it should be for an entire function; and the order ρ is given by

$$1/\rho = \lim_{k \rightarrow \infty} \frac{\log \left(k! / \Gamma\left(\frac{k+1}{3}\right) \right)}{k \log(k)} = \frac{2}{3}$$

Hence, $\rho = 3/2$.

I. 3 RELATION OF $Ah_{\nu,3}$ TO OTHER FUNCTIONS

Certain connections between the extended Airy-Hardy integrals and other functions can be derived by manipulation of the terms of the power series 1(I. 2). For example, to express $Ah_{1,3}(B)$ in terms of Bessel functions, make the change of index

$(k+1)/3 = m+1$ in 1(I.2) and write the terms in the form:

$$Ah_{1,3}(B) = \frac{\sqrt{3}}{6\pi} \sum_{m=0}^{\infty} \left\{ \left[\frac{2}{\sqrt{3}} \sin(m+1)\pi \right] \frac{\Gamma(m+1)}{(3m+2)!} B^{3m+2} + \left[\frac{2}{\sqrt{3}} \sin\left(m+\frac{2}{3}\right)\pi \right] \frac{\Gamma\left(m+\frac{2}{3}\right)}{(3m+1)!} B^{3m+1} \right. \\ \left. + \left[\frac{2}{\sqrt{3}} \sin\left(m+\frac{1}{3}\right)\pi \right] \frac{\Gamma\left(m+\frac{1}{3}\right)}{(3m)!} B^{3m} \right\}$$

The first bracket is zero; the second and third are each equal to $(-1)^m$. Hence,

$$Ah_{1,3}(B) = \frac{\sqrt{3}}{6\pi} \sum_{m=0}^{\infty} (-1)^m \left\{ \frac{\Gamma\left(m+\frac{2}{3}\right) B^{3m+1}}{(3m+1)!} + \frac{\Gamma\left(m+\frac{1}{3}\right) B^{3m}}{(3m)!} \right\} \quad 1(I.3)$$

By making use of the identity,

$$\Gamma\left(m+\frac{1}{3}\right) \Gamma\left(m+\frac{2}{3}\right) = \frac{2\pi}{\sqrt{3}} \cdot \frac{(3m)!}{3^{3m} m!}$$

the gamma functions in the numerators of 1(I.3) can be shifted to the denominator, giving

$$Ah_{1,3}(B) = \frac{1}{3} \sum_{m=0}^{\infty} \frac{(-1)^m}{m! 3^{3m}} \left\{ \frac{B^{3m+1}}{\Gamma\left(m+\frac{1}{3}\right)(3m+1)} + \frac{B^{3m}}{\Gamma\left(m+\frac{2}{3}\right)} \right\}$$

or

$$Ah_{1,3}(B) = \frac{1}{3} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left\{ \frac{(B/3)^{3m+1}}{\Gamma\left(m+\frac{4}{3}\right)} + \frac{(B/3)^{3m}}{\Gamma\left(m+\frac{2}{3}\right)} \right\} \\ Ah_{1,3}(B) = \frac{1}{3} \left(\frac{B}{3}\right)^{1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left\{ \frac{\left[\left(\frac{B}{3}\right)^{3/2}\right]^{2m+1/3}}{\Gamma\left(m+\frac{4}{3}\right)} + \frac{\left[\left(\frac{B}{3}\right)^{3/2}\right]^{2m-1/3}}{\Gamma\left(m+\frac{2}{3}\right)} \right\}$$

These two sums can now be identified with those for the Bessel functions of the first kind of orders $1/3$ and $-1/3$:

$$Ah_{1,3}(B) = \frac{1}{3} \sqrt{\frac{B}{3}} \left\{ J_{1/3}\left(2\left[\frac{B}{3}\right]^{3/2}\right) + J_{-1/3}\left(2\left[\frac{B}{3}\right]^{3/2}\right) \right\} \quad 2(I.3)$$

If the development of 1(I.2) is carried out in terms of $-B$ instead of B , we obtain:

$$Ah_{1,3}(B) = \frac{1}{3} \sqrt{\frac{-B}{3}} \left\{ I_{-1/3}\left(2\left[\frac{-B}{3}\right]^{3/2}\right) - I_{1/3}\left(2\left[\frac{-B}{3}\right]^{3/2}\right) \right\} = \frac{\sqrt{-B}}{3\pi} K_{1/3}\left(2\left[\frac{-B}{3}\right]^{3/2}\right) \quad 3(I.3)$$

Since the expansions of $J_{\pm 1/3}(z)$ and $I_{\pm 1/3}(z)$ converge for all finite z exterior to a cut, which may be placed along the negative real z -axis, the expansions 2(I.3) and 3(I.3)

converge for $|\arg(B)| < 2\pi/3$ and $|\arg(-B)| < 2\pi/3$, respectively. Hence, the entire finite B-plane is covered, and the functions $Ah_{2,3}(B)$ and $Ah_{3,3}(B)$ can be similarly expressed in terms of Bessel functions through the relation 3(I. 1).

Known relations between the Bessel functions and the confluent hypergeometric function lead to the representation:

$$Ah_{1,3}(B) = \frac{Be^{\pm i2(B/2)^{3/2}}}{3\Gamma(1/3)} \cdot {}_1F_1\left(\frac{5}{6} ; \frac{5}{3} ; \mp i4\left[\frac{B}{2}\right]^{3/2}\right) + \frac{e^{\pm i2(B/2)^{3/2}}}{3\Gamma(2/3)} \cdot {}_1F_1\left(\frac{1}{6} ; \frac{1}{3} ; \mp i4\left[\frac{B}{2}\right]^{3/2}\right) \quad 4(I. 3)$$

$Ah_{1,3}(B)$, $Ah_{2,3}(B)$, and $Ah_{3,3}(B)$ are actually generalizations of combinations of certain functions of Airy, $Ai(x)$ and $Bi(x)$, defined for real argument⁽⁴⁾. The Airy integral is defined by

$$Ai(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{-xt+(1/3)t^3} dt$$

so that the equivalence

$$Ah_{1,3}(B) = 3^{1/3} Ai(-3^{1/3}B) \quad 5(I. 3)$$

follows. In a similar fashion,

$$\left. \begin{array}{l} Ah_{2,3}(B) \\ Ah_{3,3}(B) \end{array} \right\} = \frac{1}{2} 3^{1/3} [Ai(-3^{1/3}B) \mp iBi(-3^{1/3}B)] \quad 6(I. 3)$$

where $Bi(x)$ is the Airy integral of the second kind. Both $Ai(x)$ and $Bi(x)$ [also $Ai'(x)$] are well tabulated for both positive and negative real values of x , and their zeros, turning points, maxima, and minima are known over a wide range⁽⁴⁾.

I. 4 DIFFERENTIAL EQUATION SATISFYING $Ah_{\nu,3}$

Double differentiation of the series 1(I. 2) yields a series of almost the same form as 1(I. 2) itself. Closer inspection reveals that

$$Ah_{1,3}''(B) = -\frac{B}{3} Ah_{1,3}(B) \quad 1(I. 4)$$

In terms of the Airy integral, this differential equation reads

$$Ai''(x) = xAi(x) \quad 2(I. 4)$$

I. 5 ASYMPTOTIC EXPANSIONS FOR LARGE $|B|$

The most direct method for obtaining an asymptotic series to represent $Ah_{1,3}(B)$ [hence, $Ah_{2,3}(B)$ and $Ah_{3,3}(B)$] from 3(I. 1) for large $|B|$ is that of utilizing the known expansions for the Bessel functions in the relations 2(I. 3) and 3(I. 3). For $|\arg(y)| < \pi/2$, Watson⁽⁵⁾ gives:

$$J_{\pm 1/3}(y) \sim \sqrt{\frac{\pi}{2y}} \left\{ \cos\left(y \mp \frac{\pi}{6} - \frac{\pi}{4}\right) \sum_{m=0}^{\infty} \frac{\Gamma\left(\frac{1}{2} \pm \frac{1}{3} + 2m\right) \cdot (-1)^m}{\Gamma\left(\frac{1}{2} \pm \frac{1}{3} - 2m\right) (2m)!} \cdot \frac{1}{(2y)^{2m}} \right. \\ \left. - \sin\left(y \mp \frac{\pi}{6} - \frac{\pi}{4}\right) \sum_{m=0}^{\infty} \frac{\Gamma\left(\frac{1}{2} \pm \frac{1}{3} + 2m + 1\right) \cdot (-1)^m}{\Gamma\left(\frac{1}{2} \pm \frac{1}{3} - 2m - 1\right) (2m+1)!} \cdot \frac{1}{(2y)^{2m+1}} \right\}$$

which can be written in the more abbreviated form:

$$J_{1/3}(y) \sim \sqrt{\frac{2}{\pi y}} \sum_{m=0}^{\infty} \frac{\Gamma\left(\frac{1}{2} \pm \frac{1}{3} + m\right)}{\Gamma\left(\frac{1}{2} \pm \frac{1}{3} - m\right)} \cdot \frac{1}{m! (2y)^m} \sin\left(y + \frac{m\pi}{2} + \frac{\pi}{4} \mp \frac{\pi}{6}\right) \quad 1(I.5)$$

and

$$K_{1/3}(y) \sim \sqrt{\frac{\pi}{2y}} e^{-y} \sum_{m=0}^{\infty} \frac{\Gamma\left(\frac{5}{6} + m\right)}{\Gamma\left(\frac{5}{6} - m\right)} \cdot \frac{1}{m! (2y)^m} \quad 2(I.5)$$

For large $|B|$, then, and for $|\arg(B)| < \pi/3$, relation 2(I.3) yields

$$Ah_{1,3}(B) \sim \frac{1}{\sqrt{3\pi} \sqrt{3B}} \sum_{m=0}^{\infty} \frac{1}{m! [4(B/3)^{3/2}]^m} \left\{ \frac{\Gamma\left(\frac{5}{6} + m\right)}{\Gamma\left(\frac{5}{6} - m\right)} \sin\left(y + \frac{\pi}{4} - \frac{\pi}{6} + \frac{m\pi}{2}\right) \right. \\ \left. + \frac{\Gamma\left(\frac{1}{6} + m\right)}{\Gamma\left(\frac{1}{6} - m\right)} \sin\left(y + \frac{\pi}{4} + \frac{\pi}{6} + \frac{m\pi}{2}\right) \right\}$$

or, better,

$$Ah_{1,3}(B) \sim \frac{1}{\sqrt{\pi} \sqrt{3B}} \sum_{m=0}^{\infty} \frac{(-1)^m (6m)!}{(2m)! (3m)! (4\sqrt{3B})^{3m}} \sin\left(y + \frac{(2m+1)\pi}{4}\right) \quad 3(I.5)$$

For $|\arg(-B)| < \pi/3$, relation 3(I.3) yields

$$Ah_{1,3}(B) \sim \frac{e^{-2(-B/3)^{3/2}}}{2\sqrt{\pi} \sqrt{-3B}} \sum_{m=0}^{\infty} \frac{\Gamma\left(\frac{5}{6} + m\right)}{\Gamma\left(\frac{5}{6} - m\right)} \cdot \frac{1}{m! [4(-B/3)^{3/2}]^m}$$

or

$$Ah_{1,3}(B) \sim \frac{e^{-2(-B/3)^{3/2}}}{2\sqrt{\pi} \sqrt{-3B}} \sum_{m=0}^{\infty} \frac{(-1)^m (6m)!}{(2m)! (3m)! (4\sqrt{-3B})^{3m}} \quad 4(I.5)$$

An alternative expansion, more suited than 3(I. 5) to computation when B is not real, can be derived by applying the saddlepoint method of integration to the defining integral for $Ah_{1,3}(B)$, namely,

$$Ah_{1,3}(B) = \frac{1}{2\pi i} \int_{\gamma_1} e^{Bz+z^3} dz \quad 5(I. 5)$$

This method is based upon the fact that the magnitude of the integrand of this integral, when considered as a function of z , is relatively small over all of the contour γ_1 except that portion near $z = 0$, where this magnitude assumes larger values. Consequently, almost all of the value of the integral could be obtained by integration over only this short section of contour where the integrand is large. The approximation to the true value of the integral should improve as the integrand becomes smaller along the tails of contour.

The process of integration along the contour γ_1 can be visualized as a passage over mountainous terrain that represents the magnitude of the integrand, which, for $z = re^{i\theta}$, is

$$e^{\text{Re}\{W(z)\}} = e^{rb \cos(\theta+\beta)} + r^3 \cos(3\theta)$$

where $B = be^{i\beta}$. For large r , it is clear that there are three ridges of increasing altitude in the directions 0° , $+120^\circ$, and -120° , separated by three valleys of increasing depth in the directions -60° , 60° , and 180° . Near $r = 0$, these three ridges and three valleys conflow to form two mountain passes, or "saddlepoints," at which points the terrain is flat. These points are given by

$$\frac{dW}{dz} = 0 = B + 3z^2$$

or

$$z = z_s = \pm \sqrt{-B/3} = \pm \sqrt{b/3} e^{i\frac{1}{2}(\beta+\pi)} \quad 6(I. 5)$$

In order to carry out the integration along the contour near $z = 0$, we must distort the contour from the position shown in Fig. 1a in such a way that the resultant line integral can be evaluated in terms of known functions. It would not be illogical to try, first, a pair of connected contours that pass directly through the lower and upper saddlepoints, oriented in a convenient direction. With this choice of contour in mind, let the exponent W be expanded in a Taylor series about either saddlepoint z_s :

$$W(z) = W(z_s) + W'(z_s)(z - z_s) + \frac{1}{2}W''(z_s)(z - z_s)^2 + \dots$$

or

$$W(z) = \mp 2(-B/3)^{3/2} \pm \sqrt{-3B}(z - z_s)^2 + (z - z_s)^3 \quad 7(I. 5)$$

where the upper sign refers to the saddlepoint at $z = + \sqrt{-B/3}$, and the lower sign to the saddlepoint at $z = - \sqrt{-B/3}$. We now let $z - z_s = \rho e^{i\phi}$, and choose a convenient value of ϕ for each section of the contour. With this substitution in 6(I.5), the integral for $Ah_{1,3}(B)$ becomes

$$\begin{aligned} Ah_{1,3}(B) = & \frac{1}{2\pi i} e^{2(-B/3)^{3/2}} \int_{-\infty}^{\rho_1} e^{-\sqrt{3B}\rho^2} e^{i2\phi_1} e^{\rho^3} e^{i3\phi_1} d\rho e^{i\phi_1} \\ & + \frac{1}{2\pi i} e^{-2(-B/3)^{3/2}} \int_{-\rho_2}^{\infty} e^{\sqrt{3B}\rho^2} e^{i2\phi_2} e^{\rho^3} e^{i3\phi_2} d\rho e^{i\phi_2} \end{aligned} \quad 8(I.5)$$

These two integrals represent the contributions along the lower and upper contours, respectively. The angles ϕ_1 and ϕ_2 are the inclinations of the contours, and the distances ρ_1 and ρ_2 are measured from the corresponding saddlepoints to the point of intersection of the contours. This arrangement of contours is shown in Fig. 2 for $\beta = \arg(B) = 40^\circ$.

These integrals would be simplified for evaluation if the angles ϕ_1 and ϕ_2 could be chosen to make the exponential terms involving ρ^2 negative real. This choice of contour directions corresponds to the directions of steepest descent from the "mountain passes" at the saddlepoints to the valleys below, since any other choice of ϕ_1 and ϕ_2 would make the real multipliers of ρ^2 smaller, thus indicating a more gradual decrease in altitude on the two sides of each pass. As the contours pass into the valley, they deviate from the paths of steepest descent (shown as dotted lines in Fig. 2), because of the effect of the ρ^3 term. For large $|B|$, the size of the integrand is controlled mainly by the second-degree term, so that the departure of the contours from the bottom of the valleys is not great over significant sections of the contours.

The ρ^2 -coefficients are apparently negative real, provided that

$$\left. \begin{aligned} -\frac{\pi}{2} + \frac{\beta}{2} + 2\phi_1 &= \pi \quad , \quad \text{or} \quad \phi_1 = \frac{3\pi}{4} - \frac{\beta}{4} \\ -\frac{\pi}{2} + \frac{\beta}{2} + 2\phi_2 &= 0 \quad , \quad \text{or} \quad \phi_2 = \frac{\pi}{4} - \frac{\beta}{4} \end{aligned} \right\} \quad 9(I.5)$$

and

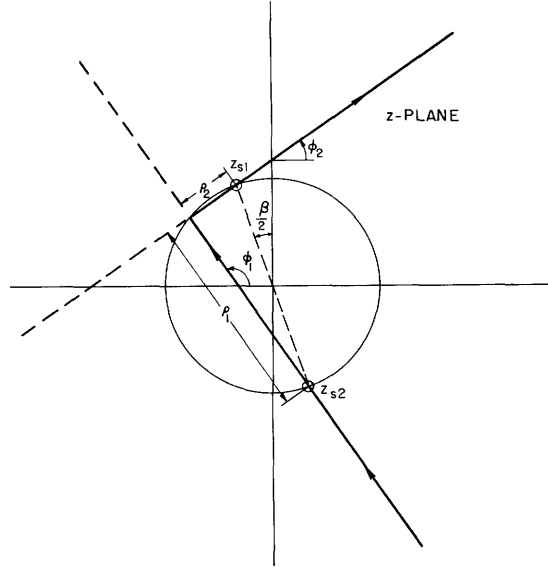


Fig. 2. L-shaped contour pertinent to the evaluation of the first extended Airy-Hardy integral by the saddlepoint method.

The evaluation of the integral is now a straightforward process. The ρ^3 -terms, being small in comparison with the ρ^2 -terms for large $|B|$, can be removed from the exponent with the rapidly convergent development,

$$e^{\rho^3} e^{i3\phi} = \sum_{m=0}^{\infty} \frac{\rho^{3m}}{m!} e^{i3m\phi}$$

The integrals 6(I.5), therefore, become

$$\begin{aligned} Ah_{1,3}(B) = & \frac{e^{-i2(B/3)^{3/2} + i\phi_1}}{2\pi i} \sum_{m=0}^{\infty} \frac{e^{i3m\phi_1}}{m!} \int_{-\infty}^{\rho_1} \rho^{3m} e^{-\sqrt{3}b} \rho^2 d\rho \\ & + \frac{e^{i2(B/3)^{3/2} + i\phi_2}}{2\pi i} \sum_{m=0}^{\infty} \frac{e^{i3m\phi_2}}{m!} \int_{-\rho_2}^{\infty} \rho^{3m} e^{-\sqrt{3}b} \rho^2 d\rho \end{aligned} \quad 10(I.5)$$

After a considerable amount of algebraic computation, an asymptotic formula for $Ah_{1,3}(B)$ for large $|B|$ and for $|\beta| < 60^\circ$ is obtained:

$$Ah_{1,3}(B) \sim \frac{1}{2\pi} \sum_{m=0}^{\infty} \frac{1}{m! (3B)^{(3m+1)/4}} \left\{ P_m(B) \sin\left(y + \frac{(3m+1)\pi}{4}\right) + i Q_m(B) \cos\left(y + \frac{(3m+1)\pi}{4}\right) \right\} \quad 11(I.5)$$

where $y = 2(B/3)^{3/2}$, and

$$\begin{aligned} P_m(B) &= \Gamma\left(\frac{3m+1}{2}\right) + \frac{(-1)^m}{2} \left\{ \Gamma\left(\frac{3m+1}{2} ; |y| [1 + \sin(\beta/2)]\right) + \Gamma\left(\frac{3m+1}{2} ; |y| [1 - \sin(\beta/2)]\right) \right\} \\ Q_m(B) &= \frac{(-1)^m}{2} \left\{ \Gamma\left(\frac{3m+1}{2} ; |y| [1 + \sin(\beta/2)]\right) - \Gamma\left(\frac{3m+1}{2} ; |y| [1 - \sin(\beta/2)]\right) \right\} \end{aligned} \quad 12(I.5)$$

The latter functions are incomplete gamma functions, defined by

$$\Gamma(x; p) = \int_0^p t^{x-1} e^{-t} dt$$

and are well tabulated⁽⁶⁾.

The formula 11(I.5) is actually a refined form of the asymptotic development 3(I.5) derived from Bessel-function expansions. Equation 3(I.5) can be obtained from 11(I.5) by extending the tails of the two sections of the L-shaped contour of Fig. 2 to infinity – that is, by letting ρ_1 and ρ_2 approach infinity so that both of the integrals in 10(I.5) have doubly infinite limits. Since the contributions to the integrals along these added portions of the contours are small, especially for large $|B|$, we would expect a significant improvement in the accuracy of the asymptotic series only for moderate values of $|B|$. When $\rho_1 \rightarrow \infty$ and $\rho_2 \rightarrow \infty$ in 10(I.5), then the incomplete gamma function becomes the complete gamma function in 11(I.5):

$$\lim_{p \rightarrow \infty} \Gamma(x; p) = \Gamma(x)$$

The equivalence of 11(I. 5) and 3(I. 5) now follows from elementary properties of the gamma functions.

When $\beta = 180^\circ$ (B negative real) the two saddlepoints lie on the real z -axis. A single contour through the right-hand saddlepoint is equivalent to the original γ_1 -contour of Fig. 1a. (The right-hand saddlepoint is chosen because the directions of steepest descent are oriented normal to the real axis for this saddlepoint, but lie along the real axis for the left-hand saddlepoint.) For this contour, $\phi = 90^\circ$; hence, $z - z_s = i\rho$, and the integral corresponding to 8(I. 5) becomes

$$Ah_{1,3}(B) = \frac{1}{2\pi} e^{-2(-B/3)^{3/2}} \int_{-\infty}^{\infty} e^{-\sqrt{-3B}\rho^2 - i\rho^3} d\rho \quad 13. (I. 5)$$

The integration is straightforward, and yields the result 4(I. 5) that would be expected from the relationship of 11(I. 5) and 3(I. 5).

II. TABLES OF THE EXTENDED AIRY-HARDY INTEGRALS

II.1 PREPARATION OF THE TABLES

The tables at the end of this report give the real and imaginary parts, magnitude, and angle (in radians) of the first extended Airy-Hardy integral, $Ah_{1,3}(B) = Ah_{1,3}(|B|e^{i\beta})$, and the real and imaginary parts of the second and third functions, $Ah_{2,3}(B)$ and $Ah_{3,3}(B)$. The magnitude and angle of $Ah_{2,3}(B)$ and $Ah_{3,3}(B)$ can be read from the appropriate tables for the magnitude and angle of $Ah_{1,3}(B)$, using the relation 3(I. 1), which can be written:

$$\left. \begin{array}{l} |Ah_{2,3}(B)| \\ |Ah_{3,3}(B)| \end{array} \right\} = |Ah_{1,3}(Be^{\pm i 120^\circ})| \quad 1(II. 1)$$

$$\arg \left\{ \begin{array}{l} Ah_{2,3}(B) \\ Ah_{3,3}(B) \end{array} \right\} = \arg Ah_{1,3}(B) \mp 60^\circ$$

All values are given to seven decimal places (last figure uncertain, particularly in the values of $\arg Ah_{1,3}(B)$) for the range of the independent variable,

$$|B| = 0(0.2)4 \quad ; \quad \beta = 0(7.5)180^\circ$$

$Ah_{1,3}(B)$ is real for $\text{Re}\{B\}$; therefore, for negative values of β , we have

$$Ah_{1,3}(\bar{B}) = \overline{Ah_{1,3}(B)} \quad 2(II. 1)$$

The values in the tables were computed by direct calculation of $\text{Re} Ah_{1,3}(B)$ and $\text{Im} Ah_{1,3}(B)$ from the series 1(I. 2). The second and third columns in each of Tables 1-25 were computed directly from the first two columns; the last four columns were computed from the first two by use of relation 3(I. 1).

Table 26 gives the coefficients of the series 1(I. 2) to eight significant figures. Table 27 gives the zeros of $Ah_{1,3}(B)$ for $\text{Re}\{B\}$. This table was taken from reference 4 (page 343), with the appropriate change of scale. Figures 3-5, which follow the tables, are plots of the functions computed along rays $\beta = \text{constant}$. The figures are self-explanatory.

The natural extension of these tables to larger values of B is through the asymptotic expansions 3(I. 5) or 11(I. 5) and 4(I. 5).

Existing tables of the Airy-Hardy integral in various forms are listed in references 4, 5, and 7 for real argument; in references 8 and 9 for complex argument.

References

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2. M. V. Cerrillo, Transient phenomena in waveguides, Technical Report 33, Research Laboratory of Electronics, M. I. T., Jan. 3, 1948.
3. M. V. Cerrillo and W. H. Kautz, Properties and tables of Lommel functions of two variables, Technical Report 138, Research Laboratory of Electronics, M. I. T. (to be published).
4. British Association Mathematical Tables, Pt. -Vol. B, The Airy Integral (Cambridge University Press, London, 1946). Tables: (i) $Ai(x)$ and $Ai'(x)$ for $x = -20(.01)2$, and first fifty zeros and turning values; 8D, δ^2 . (ii) $Bi(x)$ and $Bi'(x)$ for $x = -10(.1)2.5$, and first twenty zeros and turning values; 8D, δ^2 . (iii) $\log Ai(x)$ and $Ai'(x)/Ai(x)$ for $x = 0(.1)25(1)75$; 7-8D, δ^2 . (iv) $\log Bi(x)$ and $Bi'(x)/Bi(x)$ for $x = 0(.1)10$; 7-8D, δ^2 . (v) envelope and phase functions for both $Ai(x)$ and $Bi(x)$ for $x = -80(1)-30(.1)2.5$; 8D, δ^2 .
5. G. N. Watson, A Treatise on the Theory of Bessel Functions (Cambridge University Press, London, 2nd ed., 1948). Tables: (i) $J_{1/3}(x)$, $Y_{1/3}(x)$, $H_{1/3}(x)$, and $\text{Arg } H_{1/3}(x)$ for $x = 0(.02)16$; 7D. (ii) $J_{-1/3}(x)$ by simple computation with values in (i). (iii) Zeros of $J_{1/3}(x)$, $J_{-1/3}(x)$, and $J_{1/3}(x) \pm J_{-1/3}(x)$; 7D; see also pp. 188-190; 199; 320-324.
6. K. Pearson, Tables of the Incomplete Gamma-function (Cambridge University Press, London, 1946).
7. The Computation Laboratory, National Bureau of Standards, Tables of Bessel Functions of Fractional Order, Vols. I and II (Columbia University Press, New York, 1948, 1949).
8. Tables of the Modified Hankel Functions of Order One-third and of their Derivatives, Annals of the Computation Laboratory of Harvard University, Vol. II (Harvard University Press, Cambridge, Mass., 1945). Tables: $h_1(z)$, $h_1'(z)$, $h_2'(z)$ for $|z| = |x + jy| < 6$, with x and y increments of 0.1, and zeros of these quantities, where $h_{1,2}(z) = (Z)^{1/3} H_{1/3}^{(1),(2)}(Z)$, and $Z^3 = (2/3)z^{3/2}$.
9. P. M. Woodward and A. M. Woodward, Four-figure tables of the Airy function in the complex plane, Phil. Mag. ser. 7, 37, 236-261 (1946). Tables: Real and imaginary parts of $Ai(z)$, $Ai'(z)$, $Bi(z)$, and $Bi'(z)$ for $z = x + jy$, $x = -2.4(.2)2.4$, $y = -2.4(.2)0$; 4D, δ^2 .

TABLE 1

$$\beta = 0^\circ$$

B	Re Ah _{1,3} (B)	Im Ah _{1,3} (B)	Ah _{1,3} (B)	Arg Ah _{1,3} (B)	Re Ah _{2,3} (B)	Im Ah _{2,3} (B)	Re Ah _{3,3} (B)	Im Ah _{3,3} (B)
0	0.2461627	0	0.2461627	0	0.1230814	-0.2131832	0.1230814	0.2131832
0.2	.2709332		.2709332		.1354666	- .1915417	.1354666	.1915417
0.4	.2949706		.2949706		.1474853	- .1693994	.1474853	.1693994
0.6	.3174251		.3174251		.1587125	- .1463640	.1587125	.1463640
0.8	.3373324		.3373324		.1686662	- .1221689	.1686662	.1221689
1.0	.3536366		.3536366		.1768183	- .0966829	.1768183	.0966829
1.2	.3652247		.3652247		.1826123	- .0699208	.1826123	.0699208
1.4	.3709731		.3709731		.1854866	- .0420538	.1854866	.0420538
1.6	.3698063		.3698063		.1849031	- .0134154	.1849031	.0134154
1.8	.3607666		.3607666		.1803832	.0154959	.1803832	- .0154959
2.0	.3430916		.3430916		.1715458	.0440232	.1715458	- .0440232
2.2	.3162978		.3162978		.1581488	.0713668	.1581488	- .0713668
2.4	.2802624		.2802624		.1401312	.0966102	.1401312	- .0966102
2.6	.2353006		.2353006		.1176503	.1187599	.1176503	- .1187599
2.8	.1822270		.1822270		.0911135	.1367954	.0911135	- .1367954
3.0	.1223966		.1223966		.0611983	.1497328	.0611983	- .1497328
3.2	.0577143		.0577143		.0288572	.1566965	.0288572	- .1566965
3.4	- .0093923		.0093923		- .0046961	.1569965	- .0046961	- .1569965
3.6	- .0760450		.0760450		- .0380225	.1502077	- .0380225	- .1502077
3.8	- .1390331		.1390331		- .0695166	.1362422	- .0695166	- .1362422
4.0	- .1949801		.1949801		- .0974901	.1154106	- .0974901	- .1154106

TABLE 2

 $\beta = 7.5^\circ$

B	Re $Ah_{1,3}$ (B)	Im $Ah_{1,3}$ (B)	$Ah_{1,3}$ (B)	Arg $Ah_{1,3}$ (B)	Re $Ah_{2,3}$ (B)	Im $Ah_{2,3}$ (B)	Re $Ah_{3,3}$ (B)	Im $Ah_{3,3}$ (B)
0	0.2461627	0	0.2461627	0	0.1230814	-0.2131832	0.1230814	0.2131832
0.2	.2707294	0.0032036	.2707484	0.0118327	.1325178	- .1901309	.1382116	.1933345
0.4	.2946231	.0061177	.2946866	.0207615	.1414341	- .1667566	.1531890	.1728743
0.6	.3170689	.0083958	.3171800	.0264733	.1493141	- .1428604	.1677547	.1512562
0.8	.3371896	.0096381	.3373274	.0285758	.1556556	- .1183766	.1815341	.1280147
1.0	.3540215	.0094079	.3541465	.0265680	.1599836	- .0933724	.1940378	.1027803
1.2	.3665377	.0072572	.3666095	.0197966	.1618702	- .0680446	.2046675	.0753018
1.4	.3736810	.0027602	.3736912	.0073864	.1609549	- .0427111	.2127260	.0454713
1.6	.3744045	- .0044443	.3744309	- .0118698	.1569682	- .0177982	.2174363	.0133539
1.8	.3677223	- .0146038	.3680122	- .0396934	.1497535	.0061785	.2179687	- .0207824
2.0	.3527678	- .0277983	.3538614	- .0786381	.1392876	.0286402	.2134802	- .0564384
2.2	.3288600	- .0438850	.3317752	- .1326620	.1256956	.0489750	.2031644	- .0928600
2.4	.2955750	- .0624475	.3020997	- .2082127	.1092602	.0665752	.1863148	- .1290227
2.6	.2528207	- .0827549	.2660200	- .3163346	.0904224	.0808763	.1623982	- .1636311
2.8	.2009109	- .1037372	.2261119	- .4766297	.0697724	.0913978	.1311384	- .1951350
3.0	.1406338	- .1239859	.1874843	- .7225683	.0480302	.0977834	.0926036	- .2217693
3.2	.0733096	- .1417851	.1596161	- 1.0936041	.0260147	.0998360	.0472948	- .2416210
3.4	.0008296	- .1551812	.1551834	- 1.5654501	.0046050	.0975456	- .0037754	- .2527267
3.6	- .0743297	- .1620942	.1783239	- 2.0007447	- .0153089	.0911075	- .0590208	- .2532017
3.8	- .1491280	- .1604717	.2190669	- 2.3195708	- .0328784	.0809272	- .1162495	- .2413990
4.0	- .2200241	- .1484831	.2654390	- 2.5479465	- .0473524	.0676111	- .1726718	- .2160943

TABLE 3

 $\beta = 15^\circ$

B	Re Ah _{1,3} (B)	Im Ah _{1,3} (B)	Ah _{1,3} (B)	Arg Ah _{1,3} (B)	Re Ah _{2,3} (B)	Im Ah _{2,3} (B)	Re Ah _{3,3} (B)	Im Ah _{3,3} (B)
0	0.2461627	0	0.2461627	0	0.1230814	-0.2131832	0.1230814	0.2131832
0.2	.2701201	0.0063587	.2701950	0.0235359	.1294192	- .1891221	.1407008	.1954808
0.4	.2935746	.0121868	.2938274	.0414880	.1351624	- .1649579	.1584123	.1771447
0.6	.3159624	.0168536	.3164116	.0532900	.1397799	- .1406903	.1761824	.1575439
0.8	.3366524	.0196315	.3372243	.0582478	.1428180	- .1164255	.1938343	.1360570
1.0	.3549455	.0197186	.3554928	.0554969	.1439090	- .0923668	.2110365	.1120854
1.2	.3700719	.0162739	.3704295	.0439466	.1427811	- .0687994	.2272908	.0850733
1.4	.3811894	.0084636	.3812833	.0221996	.1392689	- .0460725	.2419205	.0545361
1.6	.3873808	- .0044782	.3874067	- .0115597	.1333201	- .0245772	.2540608	.0200990
1.8	.3876553	- .0231775	.3883476	- .0597179	.1249991	- .0047214	.2626562	- .0184561
2.0	.3809531	- .0480358	.3839697	- .1254318	.1144859	.0130962	.2664672	- .0611320
2.2	.3661591	- .0791370	.3746133	- .2128535	.1020688	.0285132	.2640902	- .1076502
2.4	.3421299	- .1161485	.3613078	- .3272782	.0881327	.0412287	.2539972	- .1573773
2.6	.3077377	- .1582226	.3460302	- .4749017	.0731398	.0510258	.2345979	- .2092484
2.8	.2619380	- .2039047	.3319467	- .6614596	.0576071	.0577886	.2043309	- .2616932
3.0	.2038661	- .2510574	.3234056	- .8887629	.0420791	.0615150	.1617870	- .3125724
3.2	.1329659	- .2968133	.3252354	-1.1496249	.0270981	.0623183	.1058678	- .3591344
3.4	.0491551	- .3375678	.3411279	-1.4261971	.0131747	.0604383	.0359803	- .3980061
3.6	- .0469778	- .3690291	.3720072	-1.6974163	.0007593	.0562031	- .0477370	- .4252322
3.8	- .1539465	- .3863399	.4158823	-1.9499865	- .0097832	.0500400	- .1441633	- .4363800
4.0	- .2691261	- .3842896	.4691560	-2.1817378	- .0181922	.0424376	- .2509338	- .4267273

TABLE 4

 $\beta = 22.5^\circ$

B	Re Ah _{1,3} (B)	Im Ah _{1,3} (B)	Ah _{1,3} (B)	Arg Ah _{1,3} (B)	Re Ah _{2,3} (B)	Im Ah _{2,3} (B)	Re Ah _{3,3} (B)	Im Ah _{3,3} (B)
0	0.2461627	0	0.2461627	0	0.1230814	-0.2131832	0.1230814	0.2131832
0.2	.2691120	0.0094167	.2692767	0.0349773	.1262259	- .1885276	.1428861	.1979442
0.4	.2918096	.0181499	.2923735	.0621177	.1287868	- .1639959	.1630229	.1821458
0.6	.3140001	.0253980	.3150256	.0807097	.1302810	- .1397686	.1837136	.1651666
0.8	.3354145	.0302388	.3367748	.0899105	.1303640	- .1160797	.2050505	.1463185
1.0	.3557531	.0316436	.3571576	.0887147	.1287750	- .0932037	.2269780	.1248473
1.2	.3746585	.0285000	.3757409	.0759231	.1253901	- .0714389	.2492684	.0999389
1.4	.3916797	.0196426	.3921719	.0501077	.1201900	- .0510891	.2714897	.0707317
1.6	.4062255	.0038933	.4062442	.0095838	.1132579	- .032444	.2929676	.0363376
1.8	.4175104	- .0198855	.4179837	- .0475928	.1047681	- .0157630	.3127424	- .0041226
2.0	.4244932	- .0527233	.4277549	- .1235700	.0949715	- .0012546	.3295217	- .0514687
2.2	.4258134	- .0954566	.4363817	- .2205288	.0841782	.0109327	.3416352	- .1063893
2.4	.4197307	- .1486226	.4452668	- .3403140	.0727378	.0207224	.3469930	- .1693450
2.6	.4040757	- .2123289	.4564655	- .4838139	.0610191	.0281144	.3430570	- .2404434
2.8	.3762215	- .2860958	.4728452	- .6501525	.0493897	.0331852	.3268318	- .3192811
3.0	.3330910	- .3686697	.4968571	- .8360540	.0381960	.0360838	.2948950	- .4047535
3.2	.2712144	- .4578074	.5321136	-1.0359688	.0277468	.0370228	.2434676	- .4948302
3.4	.1868584	- .5500376	.5809109	-1.2433094	.0182987	.0362666	.1685598	- .5863042
3.6	.0762467	- .6404071	.6449301	-1.4522944	.0100464	.0341166	.0662004	- .6745236
3.8	- .0641027	- .7222310	.7250702	-1.6593208	.0031166	.0308946	- .0672193	- .7531257
4.0	- .2369016	- .7868747	.8217628	-1.8632312	- .0024329	.0269268	- .2344686	- .8138015

TABLE 5

 $\beta = 30^\circ$

B	Re Ah _{1,3} (B)	Im Ah _{1,3} (B)	Ah _{1,3} (B)	Arg Ah _{1,3} (B)	Re Ah _{2,3} (B)	Im Ah _{2,3} (B)	Re Ah _{3,3} (B)	Im Ah _{3,3} (B)
0	0.2461627	0	0.2461627	0	0.1230814	-0.2131832	0.1230814	0.2131832
0.2	.2677169	0.0123285	.2680006	0.0460180	.1229921	- .1883524	.1447248	.2006809
0.4	.2893092	.0239336	.2902975	.0825386	.1224115	- .1638493	.1668977	.1877829
0.6	.3110333	.0339860	.3128846	.1088362	.1209676	- .1399983	.1900657	.1739843
0.8	.3330316	.0415402	.3356123	.1240925	.1184134	- .1171134	.2146182	.1586536
1.0	.3554724	.0455318	.3583766	.1273944	.1146186	- .0954891	.2408537	.1410209
1.2	.3785158	.0447730	.3811546	.1177386	.1095585	- .0753897	.2689574	.1201626
1.4	.4022659	.0379466	.4040517	.0940538	.1032982	- .0570390	.2989677	.0949856
1.6	.4267079	.0235995	.4273600	.0552496	.0959779	- .0406119	.3307300	.0642114
1.8	.4516290	.0001374	.4516290	.0003042	.0877945	- .0262273	.3638346	.0263647
2.0	.4765189	- .0341749	.4777428	- .0715952	.0789833	- .0139448	.3975355	- .0202301
2.2	.5004512	- .0812032	.5069964	- .1608580	.0698006	- .0037636	.4306506	- .0774395
2.4	.5219416	- .1429157	.5411543	- .2672646	.0605065	.0043744	.4614350	- .1472900
2.6	.5387831	- .2213267	.5824712	- .3897733	.0513506	.0105800	.4874325	- .2319067
2.8	.5478591	- .3183991	.6336620	- .5264586	.0425585	.0150084	.5053006	- .3334074
3.0	.5449370	- .4358894	.6978223	- .6746736	.0343234	.0178499	.5106136	- .4537393
3.2	.5244497	- .5751140	.7783339	- .8314423	.0267987	.0193197	.4976510	- .5944337
3.4	.4792780	- .7366134	.8788098	- .9939638	.0200946	.0196463	.4591834	- .7562597
3.6	.4005547	- .9196866	1.0031289	-1.1600373	.0142777	.0190614	.3862770	- .9387480
3.8	.2775241	-1.1217653	1.1555851	-1.3282668	.0093725	.0177898	.2681515	-1.1395551
4.0	.0975053	-1.3375982	1.3411474	-1.4980292	.0053660	.0160416	.0921393	-1.3536398

TABLE 6

 $\beta = 37.5^\circ$

B	Re $Ah_{1,3}$ (B)	Im $Ah_{1,3}$ (B)	$Ah_{1,3}$ (B)	Arg $Ah_{1,3}$ (B)	Re $Ah_{2,3}$ (B)	Im $Ah_{2,3}$ (B)	Re $Ah_{3,3}$ (B)	Im $Ah_{3,3}$ (B)
0	0.2461627	0	0.2461627	0	0.1230814	-0.2131832	0.1230814	0.2131832
0.2	.2659523	0.0150455	.2663776	0.0565118	.1197713	- .1885946	.1461810	.2036401
0.4	.2860598	.0294453	.2875713	.1025728	.1161282	- .1644880	.1699316	.1939333
0.6	.3069049	.0424891	.3098321	.1375694	.1119202	- .1412836	.1949847	.1837727
0.8	.3290112	.0533866	.3333144	.1608617	.1070181	- .1193331	.2219930	.1727197
1.0	.3529925	.0612493	.3582670	.1718039	.1013832	- .0989185	.2516093	.1601679
1.2	.3795332	.0650626	.3850696	.1697777	.0950520	- .0802518	.2844812	.1453144
1.4	.4093600	.0636459	.4142782	.1542417	.0881200	- .0634760	.3212400	.1271219
1.6	.4432051	.0556006	.4466791	.1247992	.0807250	- .0486682	.3624802	.1042688
1.8	.4817552	.0392437	.4833509	.0812803	.0730307	- .0358441	.4087245	.0750878
2.0	.5255832	.0125270	.5257325	.0238299	.0652122	- .0249646	.4603710	.0374916
2.2	.5750563	-.0270579	.5756925	-.0470179	.0574431	- .0159434	.5176132	- .0111145
2.4	.6302102	-.0825881	.6355987	-.1303058	.0498850	- .0086554	.5803251	- .0739328
2.6	.6905800	-.1578290	.7083860	-.2246866	.0426799	- .0029464	.6479001	- .1548826
2.8	.7549711	-.2573597	.7976311	-.3285332	.0359442	.0013573	.7190269	- .2587170
3.0	.8211528	-.3866944	.9076478	-.4401114	.0297665	.0044409	.7913863	- .3911354
3.2	.8854506	-.5523870	1.0436255	-.5577708	.0242061	.0064921	.8612444	- .5588790
3.4	.9422106	-.7620947	1.2118371	-.6801061	.0192948	.0076942	.9229158	- .7697889
3.6	.9831031	-1.0245675	1.4199402	-.8060481	.0150386	.0082204	.9680646	-1.0327880
3.8	.9962302	-1.3495126	1.6773964	-.9348805	.0114217	.0082293	.9848085	-1.3577419
4.0	.9649986	-1.7472614	1.9960322	-1.0661950	.0084104	.0078614	.9565882	-1.7551228

TABLE 7

$$\beta = 45^\circ$$

B	Re $Ah_{1,3}$ (B)	Im $Ah_{1,3}$ (B)	$Ah_{1,3}$ (B)	Arg $Ah_{1,3}$ (B)	Re $Ah_{2,3}$ (B)	Im $Ah_{2,3}$ (B)	Re $Ah_{3,3}$ (B)	Im $Ah_{3,3}$ (B)
0	0.2461627	0	0.2461627	0	0.1230814	-0.2131832	0.1230814	0.2131832
0.2	.2638423	0.0175193	.2644233	0.0663032	.1166149	- .1892466	.1472273	.2067658
0.4	.2820635	.0345738	.2841745	.1219661	.1100169	- .1658772	.1720466	.2004510
0.6	.3014900	.0506937	.3057222	.1665857	.1032128	- .1435393	.1982771	.1942330
0.8	.3229203	.0653916	.3294747	.1997989	.0961846	- .1225888	.2267357	.1879804
1.0	.3472916	.0781413	.3559740	.2213163	.0889601	- .1032807	.2583314	.1814220
1.2	.3756869	.0883436	.3859342	.2309562	.0816025	- .0857783	.2940843	.1741220
1.4	.4093464	.0952800	.4202889	.2286894	.0741988	- .0701639	.3351475	.1654439
1.6	.4496825	.0980490	.4602478	.2146805	.0668486	- .0564502	.3828339	.1544992
1.8	.4982989	.0954827	.5073645	.1893224	.0596548	- .0445926	.4386440	.1400753
2.0	.5570116	.0860378	.5636173	.1532520	.0527157	- .0345013	.5042958	.1205391
2.2	.6278710	.0676544	.6315054	.1073380	.0461189	- .0260528	.5817520	.0937072
2.4	.7131783	.0375742	.7141674	.0526370	.0399369	- .0191000	.6732415	.0566742
2.6	.8154941	-.0078954	.8155324	-.0096814	.0342245	- .0134828	.7812696	.0055874
2.8	.9376237	-.0736830	.9405144	-.0784236	.0290193	- .0090351	.9086044	-.0646479
3.0	1.0825685	-.1663434	1.0952738	-.1524637	.0243403	- .0055925	1.0582283	-.1520906
3.2	1.2534210	-.2945567	1.2875666	-.2308141	.0201903	- .0029975	1.2332306	-.2915592
3.4	1.4531680	-.4697682	1.5272130	-.3126680	.0165580	- .0011032	1.4366100	-.4686650
3.6	1.6843609	-.7070015	1.8267246	-.3974109	.0134204	.0002237	1.6709405	-.7072257
3.8	1.9485812	-1.0258887	2.2021390	-.4846064	.0107454	.0011013	1.9378357	-1.0269900
4.0	2.2456100	-1.4519631	2.6741282	-.5739661	.0084945	.0016321	2.3774082	-1.4535952

TABLE 8

 $\beta = 52.5^\circ$

B	Re Ah _{1,3} (B)	Im Ah _{1,3} (B)	Ah _{1,3} (B)	Arg Ah _{1,3} (B)	Re Ah _{2,3} (B)	Im Ah _{2,3} (B)	Re Ah _{3,3} (B)	Im Ah _{3,3} (B)
0	0.2461627	0	0.2461627	0	0.1230814	- .2131832	0.1230814	0.2131832
0.2	.2614179	0.0197039	.2621594	0.0752307	.1135724	- .1902949	.1478455	.2099988
0.4	.2773471	.0391948	.2801029	.1403908	.1041489	- .1679806	.1731981	.2071754
0.6	.2947335	.0583175	.3004476	.1953422	.0948956	- .1466965	.1998379	.2050140
0.8	.3144889	.0769727	.3237716	.2400362	.0858941	- .1267780	.2285948	.2037507
1.0	.3376723	.0951050	.3508098	.2745371	.0772209	- .1084515	.2604514	.2035566
1.2	.3655185	.1126864	.3824944	.2990467	.0689447	- .0918484	.2965737	.2045348
1.4	.3994806	.1296935	.4200061	.3139202	.0611244	- .0770196	.3383562	.2067132
1.6	.4412877	.1460785	.4648374	.3196742	.0538069	- .0639517	.3874808	.2100302
1.8	.4930225	.1617308	.5188719	.3169784	.0470267	- .0525803	.4459958	.2143112
2.0	.5572231	.1764261	.5844859	.3066309	.0408053	- .0428047	.5164178	.2192309
2.2	.6370148	.1897593	.6646777	.2895182	.0351514	- .0344989	.6018635	.2242582
2.4	.7362808	.2010514	.7632373	.2665649	.0300619	- .0275221	.7062189	.2285735
2.6	.8598802	.2092239	.8849682	.2386794	.0255235	- .0217271	.8343567	.2309510
2.8	1.0139289	.2126227	1.0359827	.2067064	.0215136	- .0169670	.9924152	.2295897
3.0	1.2061589	.2087740	1.2240938	.1713916	.0180030	- .0131003	1.1881558	.2218743
3.2	1.4463788	.1940429	1.4593370	.1333614	.0149569	- .0099943	1.4314219	.2040372
3.4	1.7470647	.1631550	1.7546665	.0931181	.0123370	- .0075275	1.7347277	.1706824
3.6	2.1241154	.1085249	2.1268860	.0510474	.0101031	- .0055910	2.1140122	.1141159
3.8	2.5978214	.0193145	2.5978932	.0074348	.0082147	- .0040892	2.5896067	.0234037
4.0	3.1941015	- .1198895	3.1963507	- .0375171	.0066316	- .0029391	3.1874698	- .1169504

TABLE 9

 $\beta = 60^\circ$

B	Re $Ah_{1,3}$ (B)	Im $Ah_{1,3}$ (B)	$Ah_{1,3}$ (B)	Arg $Ah_{1,3}$ (B)	Re $Ah_{2,3}$ (B)	Im $Ah_{2,3}$ (B)	Re $Ah_{3,3}$ (B)	Im $Ah_{3,3}$ (B)
0	0.2461627	0	0.2461627	0	0.1230814	-0.2131832	0.1230814	0.2131832
0.2	.2587176	0.0215562	.2596141	0.0831274	.1106906	- .1917216	.1480270	.2132779
0.4	.2719683	.0431796	.2753747	.1574528	.0985896	- .1707621	.1733788	.2139416
0.6	.2866763	.0650430	.2939624	.2231093	.0870093	- .1507045	.1996670	.2157474
0.8	.3036863	.0874369	.3160231	.2803363	.0761206	- .1318447	.2275657	.2192816
1.0	.3239467	.1107744	.3423630	.3294876	.0660399	- .1143845	.2579068	.2251589
1.2	.3485411	.1356012	.3739901	.3710344	.0568364	- .0984436	.2917046	.2340448
1.4	.3787317	.1626121	.4121655	.4055575	.0485397	- .0840732	.3301920	.2466852
1.6	.4160180	.1926757	.4584702	.4337295	.0411469	- .0712685	.3748710	.2639443
1.8	.4622135	.2268713	.5148902	.4562900	.0346304	- .0599816	.4275830	.2868530
2.0	.5195473	.2665391	.5839285	.4740106	.0289440	- .0501325	.4906033	.3166716
2.2	.5907949	.3133496	.6687500	.4876602	.0240288	- .0416191	.5667662	.3549686
2.4	.6794504	.3693969	.7733737	.4979704	.0198181	- .0343260	.6596323	.4037228
2.6	.7899503	.4373239	.9029251	.5056102	.0162416	- .0281313	.7737088	.4654551
2.8	.9279694	.5204887	1.0639717	.5111655	.0132282	- .0229120	.9147411	.5434007
3.0	1.1008095	.6231868	1.2649677	.5151327	.0107091	- .0185487	1.0901003	.6417356
3.2	1.3179151	.7509466	1.5168458	.5179167	.0086187	- .0149280	1.3092964	.7658747
3.4	1.5915607	.9109246	1.8338072	.5198380	.0068965	- .0119451	1.5846642	.9228697
3.6	1.9377671	1.1124340	2.2343792	.5211428	.0054874	- .0095045	1.9322796	1.1219385
3.8	2.3775328	1.3676552	2.7428348	.5220156	.0043423	- .0075211	2.3731905	1.3751762
4.0	2.9384911	1.6925923	3.3911058	.5225910	.0034176	- .0059194	2.9350735	1.6985118

TABLE 10

 $\beta = 67.5^\circ$

B	Re $Ah_{1,3}$ (B)	Im $Ah_{1,3}$ (B)	$Ah_{1,3}$ (B)	Arg $Ah_{1,3}$ (B)	Re $Ah_{2,3}$ (B)	Im $Ah_{2,3}$ (B)	Re $Ah_{3,3}$ (B)	Im $Ah_{3,3}$ (B)
0	0.2461627	0	0.2461627	0	0.1230814	-0.2131832	0.1230814	0.2131832
0.2	.2557870	0.0230386	.2568225	0.0898270	.1080140	- .1935040	.1477730	.2165426
0.4	.2660183	.0464063	.2700357	.1727096	.0934010	- .1741859	.1726172	.2205922
0.6	.2774663	.0705576	.2862969	.2490147	.0795951	- .1555302	.1978712	.2260879
0.8	.2907507	.0960935	.3062188	.3191998	.0668459	- .1377755	.2239048	.2338690
1.0	.3065108	.1237792	.3305604	.3838063	.0553113	- .1211010	.2511995	.2448802
1.2	.3254192	.1545730	.3602644	.4434456	.0450706	- .1056321	.2803485	.2602051
1.4	.3481970	.1896685	.3965038	.4987774	.0361388	- .0914451	.3120582	.2811136
1.6	.3756319	.2305531	.4407426	.5504859	.0284803	- .0785740	.3471516	.3091271
1.8	.4085968	.2790881	.4948146	.5995270	.0220226	- .0670165	.3865742	.3461046
2.0	.4480684	.3376155	.5610254	.6457315	.0166674	- .0567408	.4314011	.3943563
2.2	.4951450	.4091000	.6422861	.6905270	.0123012	- .0476914	.4828438	.4567914
2.4	.5510599	.4973167	.7422877	.7341798	.0088038	- .0397954	.5422560	.5371122
2.6	.6171878	.6070986	.8657306	.7771574	.0060545	- .0329675	.6111333	.6400662
2.8	.6950381	.7446619	1.0186262	.8198527	.0039371	- .0271149	.6911011	.7717768
3.0	.7862267	.9180360	1.2086945	.8625854	.0023437	- .0221413	.7838830	.9401772
3.2	.8924124	1.1376291	1.4458907	.9056097	.0011768	- .0179502	.8912355	1.1555793
3.4	1.0151792	1.4169770	1.7431043	.9491217	.0003505	- .0144479	1.0148287	1.4314249
3.6	1.1558334	1.7737303	2.1170900	.9932692	- .0002096	- .0115451	1.1560430	1.7852754
3.8	1.3150716	2.2309633	2.5897125	1.0381602	- .0005660	- .0091587	1.3156376	2.2401221
4.0	1.4924528	2.8189051	3.1896146	1.0838718	- .0007705	- .0072127	1.4932234	2.8261178

TABLE 11

 $\beta = 75^\circ$

B	Re $Ah_{1,3}$ (B)	Im $Ah_{1,3}$ (B)	$Ah_{1,3}$ (B)	Arg $Ah_{1,3}$ (B)	Re $Ah_{2,3}$ (B)	Im $Ah_{2,3}$ (B)	Re $Ah_{3,3}$ (B)	Im $Ah_{3,3}$ (B)
0	0.2461627	0	0.2461627	0	0.1230814	-0.2131832	0.1230814	0.2131832
0.2	.2526781	0.0241197	.2538267	0.0951677	.1055849	- .1956147	.1470933	.2197345
0.4	.2596190	.0487712	.2641603	.1856927	.0886454	- .1782160	.1709735	.2269872
0.6	.2673492	.0745966	.2775613	.2721024	.0727022	- .1611546	.1946470	.2357511
0.8	.2761637	.1023687	.2945263	.3549788	.0580727	- .1445927	.2180909	.2469614
1.0	.2862818	.1330106	.3156725	.4349403	.0449637	- .1286821	.2413182	.2616927
1.2	.2978362	.1676235	.3417661	.5126205	.0334850	- .1135590	.2643512	.2811826
1.4	.3108524	.2075243	.3737587	.5886469	.0236643	- .0993400	.2871881	.3068644
1.6	.3252172	.2542943	.4128339	.6636199	.0154630	- .0861177	.3097542	.3404120
1.8	.3406308	.3098392	.4604668	.7380961	.0087909	- .0739589	.3318399	.3837982
2.0	.3565379	.3764634	.5185017	.8125750	.0035212	- .0629038	.3530167	.4393673
2.2	.3720288	.4569584	.5892507	.8874917	- .0004971	- .0529665	.3725259	.5099250
2.4	.3857021	.5547072	.6756228	.9632144	- .0034274	- .0441363	.3891294	.5988434
2.6	.3954736	.6738056	.7812896	1.0400460	- .0054358	- .0363807	.4009094	.7101863
2.8	.3983154	.8191984	.9109013	1.1182287	- .0066850	- .0296490	.4050005	.8488474
3.0	.3898999	.9968280	1.0703681	1.1979506	- .0073269	- .0238756	.3972266	1.0207035
3.2	.3641176	1.2137887	1.2672271	1.2793538	- .0074992	- .0189841	.3716169	1.2327728
3.4	.3124292	1.4784732	1.5111238	1.3625414	- .0070736	- .0148913	.3197528	1.4933645
3.6	.2229952	1.8006895	1.8144447	1.4475848	- .0069039	- .0115106	.2298992	1.8122001
3.8	.0795184	2.1917100	2.1931520	1.5345308	- .0063265	- .0087551	.0858449	2.2004652
4.0	-.1402926	2.6641965	2.6678878	1.6234062	- .0056607	- .0065404	- .1346319	2.6707368

TABLE 12

 $\beta = 82.5^\circ$

B	Re $Ah_{1,3}$ (B)	Im $Ah_{1,3}$ (B)	$Ah_{1,3}$ (B)	Arg $Ah_{1,3}$ (B)	Re $Ah_{2,3}$ (B)	Im $Ah_{2,3}$ (B)	Re $Ah_{3,3}$ (B)	Im $Ah_{3,3}$ (B)
0	0.2461627	0	0.2461627	0	0.1230814	-0.2131832	0.1230814	0.2131832
0.2	.2494480	0.0247764	.2506754	0.0990002	.1034421	- .1980223	.1460059	.2227987
0.4	.2529169	.0501984	.2578505	.1959315	.0843867	- .1828140	.1685303	.2330124
0.6	.2566442	.0769753	.2679393	.2913927	.0663951	- .1675675	.1902491	.2445428
0.8	.2605762	.1058918	.2812704	.3859903	.0498364	- .1523470	.2107900	.2582387
1.0	.2645141	.1378161	.2982633	.4803188	.0349744	- .1372597	.2295397	.2750758
1.2	.2680865	.1737107	.3194461	.5749434	.0219741	- .1224434	.2461125	.2961541
1.4	.2707108	.2146411	.3454781	.6703841	.0109118	- .1080522	.2597990	.3226932
1.6	.2715395	.2617826	.3771788	.7671057	.0017854	- .0942440	.2697541	.3560266
1.8	.2693902	.3164219	.4155646	.8655109	- .0054734	.0811685	.2748636	.3975904
2.0	.2626542	.3799472	.4618951	.9659391	- .0109861	.0689577	.2736403	.4489049
2.2	.2491811	.4538234	.5177325	1.0686669	- .0149142	.0577189	.2640953	.5115423
2.4	.2261349	.5395427	.5850157	1.1739138	- .0174468	.0475294	.2435817	.5870721
2.6	.1898178	.6385392	.6661555	1.2818471	- .0187882	.0384351	.2086060	.6769743
2.8	.1354579	.7520541	.7641559	1.3925898	- .0191476	.0304500	.1546055	.7825040
3.0	.0569600	.8809283	.8827679	1.5062271	- .0187292	.0235581	.0756892	.9044864
3.2	-.0533812	1.0252991	1.0266878	1.6228134	- .0177254	.0177171	- .0356559	1.0430162
3.4	-.2051988	1.1841630	1.2018105	1.7423784	- .0163108	.0128627	- .1888881	1.1970256
3.6	-.4103883	1.3547625	1.4155564	1.8649324	- .0146384	.0089136	- .3957499	1.3636760
3.8	-.6834347	1.5317401	1.6772927	1.9904705	- .0128376	.0057768	- .6705971	1.5375169
4.0	-1.0416868	1.7059911	1.9988790	2.1189771	- .0110133	.0033529	-1.0306734	1.7093440

TABLE 13

 $\beta = 90^\circ$

B	Re Ah _{1,3} (B)	Im Ah _{1,3} (B)	Ah _{1,3} (B)	Arg Ah _{1,3} (B)	Re Ah _{2,3} (B)	Im Ah _{2,3} (B)	Re Ah _{3,3} (B)	Im Ah _{3,3} (B)
0	0.2461627	0	0.2461627	0	0.1230814	-0.2131832	0.1230814	0.2131832
0.2	.2461572	0.0249949	.2474229	0.1011935	.1016219	- .2006905	.1445352	.2256854
0.4	.2460736	.0506462	.2512315	.2029828	.0806919	- .1879361	.1653817	.2385823
0.6	.2457077	.0776096	.2576733	.3059443	.0607583	- .1747602	.1849494	.2523697
0.8	.2447072	.1065380	.2668931	.4106207	.0422165	- .1611057	.2024907	.2676437
1.0	.2425545	.1380750	.2791010	.5175047	.0253867	- .1470072	.2171679	.2850822
1.2	.2385426	.1728426	.2945796	.6270312	.0105101	- .1325752	.2280324	.3054178
1.4	.2317438	.2114208	.3136940	.7395714	- .0022519	- .1179784	.2339957	.3293992
1.6	.2209737	.2543149	.3369057	.8554328	- .0128180	- .1034253	.2337917	.3577401
1.8	.2047498	.3019076	.3647885	.9748604	- .0211837	- .0891459	.2259335	.3910535
2.0	.1812480	.3543909	.3980499	1.0980424	- .0274151	- .0753740	.2086631	.4297649
2.2	.1482607	.4116741	.4375578	1.2251160	- .0316409	- .0623309	.1799016	.4740050
2.4	.1031606	.4732595	.4843724	1.3561748	- .0340416	- .0502130	.1372022	.5234725
2.6	.0428792	.5380823	.5397881	1.4912754	- .0348378	- .0391809	.0777170	.5772632
2.8	- .0360890	.6043069	.6053836	1.6304452	- .0342769	- .0293525	- .0018121	.6336594
3.0	- .1376430	.6690740	.6830854	1.7736873	- .0326202	- .0208000	- .1050228	.6898740
3.2	- .2659692	.7281953	.7752470	1.9209868	- .0301307	- .0135485	- .2358385	.7417438
3.4	- .4253484	.7757943	.8847475	2.0723140	- .0270615	- .0075793	- .3982869	.7833736
3.6	- .6198411	.8038997	1.0151146	2.2276294	- .0236464	- .0028342	- .5961946	.8067338
3.8	- .8528079	.8020036	1.1706798	2.3868857	- .0200927	.0007780	- .8327152	.8012255
4.0	-1.1262168	.7566149	1.3567720	2.5500309	- .0165754	.0033737	-1.1096413	.7532412

TABLE 14

 $\beta = 97.5^\circ$

B	Re Ah _{1,3} (B)	Im Ah _{1,3} (B)	Ah _{1,3} (B)	Arg Ah _{1,3} (B)	Re Ah _{2,3} (B)	Im Ah _{2,3} (B)	Re Ah _{3,3} (B)	Im Ah _{3,3} (B)
0	0.2461627	0	0.2461627	0	0.1230814	-0.2131832	0.1230814	0.2131832
0.2	.2428678	0.0247709	.2441278	0.1016418	.1001878	- .2035607	.1427111	.2283495
0.4	.2392543	.0501091	.2444454	.2064543	.0776312	- .1935306	.1616231	.2436396
0.6	.2348953	.0765174	.2470440	.3149110	.0558999	- .1827157	.1789954	.2592330
0.8	.2292408	.1044197	.2519024	.4274203	.0353460	- .1709384	.1938948	.2753581
1.0	.2216099	.1341451	.2590480	.5443227	.0163292	- .1581243	.2052807	.2922694
1.2	.2111839	.1659033	.2685564	.6658905	- .0008271	- .1443105	.2120110	.3102138
1.4	.1970003	.1997511	.2805524	.7923315	- .0158506	- .1296321	.2128508	.3293833
1.6	.1779531	.2355486	.2952126	.9237949	- .0285313	- .1143064	.2064844	.3498550
1.8	.1528009	.2729041	.3127695	1.0603787	- .0387329	- .0986133	.1915338	.3715174
2.0	.1201877	.3111085	.3335170	1.2021377	- .0463992	- .0828751	.1665869	.3939836
2.2	.0786818	.3490594	.3578174	1.3490908	- .0515570	- .0674341	.1302388	.4164935
2.4	.0268394	.3851772	.3861112	1.5012281	- .0543150	- .0526315	.0811544	.4378088
2.6	- .0367018	.4173174	.4189282	1.6585175	- .0548574	- .0387869	.0181556	.4561042
2.8	- .1130896	.4426852	.4569020	1.8209098	- .0534341	- .0261801	- .0596555	.4688653
3.0	- .2030793	.4577633	.5007878	1.9883436	- .0503475	- .0150368	- .1527318	.4728001
3.2	- .3068017	.4582642	.5514829	2.1607485	- .0459360	- .0055181	- .2608657	.4637823
3.4	- .4234745	.4391291	.6100533	2.3380483	- .0405571	.0022861	- .3829173	.4368429
3.6	- .5510544	.3945930	.6777644	2.5201632	- .0345690	.0083579	- .5164854	.3862352
3.8	- .6858356	.3183492	.7561195	2.7070118	- .0283139	.0127482	- .6575218	.3056009
4.0	- .8220071	.2038450	.8469052	2.8985119	- .0221029	.0155704	- .7999042	.7882745

TABLE 15

 $\beta = 105^\circ$

B	Re $Ah_{1,3}$ (B)	Im $Ah_{1,3}$ (B)	$Ah_{1,3}$ (B)	Arg $Ah_{1,3}$ (B)	Re $Ah_{2,3}$ (B)	Im $Ah_{2,3}$ (B)	Re $Ah_{3,3}$ (B)	Im $Ah_{3,3}$ (B)
0	0.2461627	0	0.2461627	0	0.1230814	-0.2131832	0.1230814	0.2131832
0.2	.2396417	0.0241101	.2408515	0.1002714	.0990749	- .2066414	.1405668	.2307515
0.4	.2326179	.0486167	.2376440	.2060324	.0752765	- .1995330	.1573414	.2481497
0.6	.2245282	.0738065	.2363479	.3175913	.0519514	- .1913981	.1725768	.2652047
0.8	.2147460	.0998369	.2368190	.4351812	.0294184	- .1818968	.1853276	.2817338
1.0	.2025871	.1267202	.2389551	.5589660	.0080375	- .1708122	.1945496	.2975325
1.2	.1873210	.1543030	.2426903	.6890472	- .0118085	- .1580518	.1991296	.3123547
1.4	.1681899	.1822412	.2479913	.8254739	- .0297345	- .1436467	.1979244	.3258879
1.6	.1444366	.2099736	.2548545	.9682512	- .0453756	- .1277472	.1898122	.3377207
1.8	.1153446	.2366950	.2633038	1.1173494	- .0584107	- .1106131	.1737553	.3473081
2.0	.0802917	.2613334	.2733896	1.2727118	- .0685846	- .0925996	.1488763	.3539330
2.2	.0388173	.2825340	.2851881	1.4342612	- .0757276	- .0741376	.1145449	.3566826
2.4	- .0092941	.2986567	.2988013	1.6019060	- .0797715	- .0557108	.0704774	.3543674
2.6	- .0639155	.3077920	.3143582	1.7755444	- .0807595	- .0378280	.0168440	.3456200
2.8	- .1244675	.3078023	.3320158	1.9550684	- .0788499	- .0209949	- .0456176	.3287973
3.0	- .1898022	.2963978	.3519610	2.1403668	- .0743131	- .0056841	- .1154890	.3020819
3.2	- .2580856	.2712514	.3744136	2.3313271	- .0675207	.0076929	- .1905649	.2635585
3.4	- .3266932	.2301629	.3996291	2.5278374	- .0589284	.0188095	- .2677648	.2113535
3.6	- .3921304	.1712746	.4279033	2.7297869	- .0490530	.0274440	- .3430774	.1438306
3.8	- .4499978	.0933409	.4595765	2.9370677	- .0384443	.0334925	- .4115534	.0598484
4.0	- .4950236	- .0039514	.4950394	3.1495748	- .0276560	.0369738	- .4673676	- .0409252

TABLE 16

 $\beta = 112.5^\circ$

B	Re Ah _{1,3} (B)	Im Ah _{1,3} (B)	Ah _{1,3} (B)	Arg Ah _{1,3} (B)	Re Ah _{2,3} (B)	Im Ah _{2,3} (B)	Re Ah _{3,3} (B)	Im Ah _{3,3} (B)
0	0.2461627	0	0.2461627	0	0.1230814	-0.2131832	0.1230814	0.2131832
0.2	.2365384	0.0230275	.2376566	0.0970463	.0983993	- .2098292	.1381391	.2328568
0.4	.2263080	.0462284	.2309814	.2014999	.0736984	- .2058638	.1526096	.2520922
0.6	.2148691	.0696517	.2258763	.3134711	.0490637	- .2007400	.1658054	.2703918
0.8	.2016310	.0932058	.2221314	.4330017	.0246894	- .1939900	.1769416	.2871958
1.0	.1860293	.1166515	.2195780	.5600793	.0008711	- .1852361	.1851582	.3018877
1.2	.1675470	.1395964	.2180806	.6946463	- .0220068	- .1742060	.1895537	.3138024
1.4	.1457423	.1614905	.2175316	.8366116	- .0434886	- .1615466	.1892309	.3222371
1.6	.1202830	.1816284	.2173460	.9858591	- .0630704	- .1448375	.1833534	.3264660
1.8	.0909863	.1991576	.2189572	1.1422559	- .0802275	- .1266011	.1712138	.3257588
2.0	.0578630	.2130985	.2208146	1.3056575	- .0944469	- .1063066	.1523099	.3194051
2.2	.0211631	.2223756	.2233803	1.4759137	- .1052614	- .0843680	.1264245	.3067436
2.4	- .0185796	.2258647	.2266276	1.6528713	- .1122859	- .0613344	.0937064	.2871992
2.6	- .0605096	.2224566	.2305392	1.8363772	- .1152521	- .0378700	.0547425	.2603266
2.8	- .1034227	.2111367	.2351063	2.0262811	- .1140391	- .0147258	.0106164	.2258625
3.0	- .1457561	.1910817	.2403270	2.2224356	- .1086980	.0072964	- .0370580	.1837854
3.2	- .1856025	.1617690	.2462063	2.4246981	- .0994679	.0273886	- .0861347	.1343805
3.4	- .2207555	.1230938	.2527549	2.6329313	- .0867794	.0447847	- .1339760	.0783091
3.6	- .2487895	.0754873	.2599895	2.8470026	- .0712470	.0588116	- .1775425	.0166757
3.8	- .2671824	.0200244	.2679317	3.0667858	- .0536458	.0689372	- .2135365	- .0489127
4.0	- .2734790	- .0414910	.2766085	3.2921599	- .0348768	.0748139	- .2386022	- .1163049

TABLE 17

 $\beta = 120^\circ$

B	Re $Ah_{1,3}$ (B)	Im $Ah_{1,3}$ (B)	$Ah_{1,3}$ (B)	Arg $Ah_{1,3}$ (B)	Re $Ah_{2,3}$ (B)	Im $Ah_{2,3}$ (B)	Re $Ah_{3,3}$ (B)	Im $Ah_{3,3}$ (B)
0	0.2461627	0	0.2461627	0	0.1230814	-0.2131832	0.1230814	0.2131832
0.2	.2336133	0.0215467	.2346048	0.0919720	.0981467	- .2130884	.1354666	.2346350
0.4	.2204468	.0430263	.2246064	.1927545	.0729615	- .2124257	.1474853	.2554520
0.6	.2061112	.0642671	.2158983	.3022542	.0473986	- .2106311	.1587125	.2748982
0.8	.1901345	.0849848	.2082632	.4203330	.0214682	- .2071537	.1686662	.2921384
1.0	.1721390	.1047877	.2015249	.5468204	- .0046793	- .2014706	.1768183	.3062583
1.2	.1518594	.1231865	.1955408	.6815257	- .0307529	- .1931073	.1826123	.3162939
1.4	.1291629	.1396092	.1901941	.8242453	- .0563237	- .1816630	.1854865	.3212721
1.6	.1040696	.1534231	.1853891	.9747709	- .0808335	- .1668385	.1849031	.3202616
1.8	.0767718	.1639644	.1810476	1.1328924	- .1036114	- .1484685	.1803833	.3124330
2.0	.0476477	.1705746	.1771045	1.2984031	- .1238981	- .1265514	.1715458	.2971260
2.2	.0172690	.1726443	.1735058	1.4711015	- .1408798	- .1012775	.1581489	.2739219
2.4	- .0136013	.1696623	.1702066	1.6507921	- .1537325	- .0730521	.1401312	.2427144
2.6	- .0440239	.1612681	.1671691	1.8372889	- .1616742	- .0425082	.1176503	.2037763
2.8	- .0729116	.1473043	.1643613	2.0304137	- .1640250	- .0105089	.0911135	.1578132
3.0	- .0990733	.1278657	.1617565	2.2299958	- .1602716	.0218671	.0611983	.1059986
3.2	- .1212746	.1033393	.1593315	2.4358749	- .1501318	.0533572	.0288571	.0499820
3.4	- .1383110	.0744313	.1570667	2.6478972	- .1336149	.0825652	- .0046961	- .0081340
3.6	- .1490949	.0421754	.1549453	2.8659189	- .1110724	.1080323	- .0380225	- .0658569
3.8	- .1527475	.0079180	.1529526	3.0898018	- .0832309	.1283242	- .0695165	- .1204062
4.0	- .1486936	- .0267236	.1510759	3.3194156	- .0512035	.1421342	- .0974900	- .1688577

TABLE 18

 $\beta = 127.5^\circ$

B	Re Ah _{1,3} (B)	Im Ah _{1,3} (B)	Ah _{1,3} (B)	Arg Ah _{1,3} (B)	Re Ah _{2,3} (B)	Im Ah _{2,3} (B)	Re Ah _{3,3} (B)	Im Ah _{3,3} (B)
0	0.2461627	0	0.2461627	0	0.1230814	-0.2131832	0.1230814	0.2131832
0.2	.2309171	0.0196984	.2317558	0.0850989	.0983268	- .2163620	.1325903	.2360603
0.4	.2151325	.0391072	.2186581	.1798185	.0731190	- .2191027	.1420134	.2582099
0.6	.1983778	.0578796	.2066490	.2838844	.0471144	- .2209080	.1512634	.2787876
0.8	.1803449	.0756134	.1955548	.3970079	.0200969	- .2212205	.1602480	.2968338
1.0	.1608547	.0918637	.1852382	.5188962	- .0080085	- .2194319	.1688633	.3112955
1.2	.1398634	.1061614	.1755905	.6492593	- .0371205	- .2148981	.1769840	.3210596
1.4	.1174663	.1180355	.1665254	.7878150	- .0669837	- .2069618	.1844501	.3249973
1.6	.0938978	.1270393	.1579740	.9342922	- .0971533	- .1949823	.1910511	.3220216
1.8	.0695260	.1327796	.1498809	1.0884322	- .1269824	- .1783753	.1965084	.3111549
2.0	.0448407	.1349467	.1422016	1.2499898	- .1556172	- .1566600	.2004580	.2916067
2.2	.0204342	.1333430	.1348997	1.4187342	- .1820013	- .1295156	.2024355	.2628586
2.4	-.0030258	.1279097	.1279455	1.5944475	- .2048944	- .0968420	.2018686	.2247517
2.6	-.0248297	.1187462	.1213144	1.7769252	- .2229078	- .0588254	.1980782	.1775717
2.8	-.0442666	.1061236	.1149859	1.9659763	- .2345611	- .0160016	.1902945	.1221253
3.0	-.0606679	.0904871	.1089426	2.1614199	- .2383597	.0306876	.1776918	.0597995
3.2	-.0734531	.0724474	.1031697	2.3630875	- .2328974	.0798520	.1594442	-.0074046
3.4	-.0821744	.0527608	.0976542	2.5708203	- .2169801	.1296329	.1348057	-.0768721
3.6	-.0865558	.0322959	.0923847	2.7844697	- .1897687	.1777144	.1032128	-.1454185
3.8	-.0865243	.0119901	.0873511	3.0038951	- .1509328	.2213745	.0644086	-.2093844
4.0	-.0822291	-.0072028	.0825440	3.2289638	- .1008072	.2575853	.0185781	-.2647880

TABLE 19

 $\beta = 135^\circ$

B	Re $Ah_{1,3}$ (B)	Im $Ah_{1,3}$ (B)	$Ah_{1,3}$ (B)	Arg $Ah_{1,3}$ (B)	Re $Ah_{2,3}$ (B)	Im $Ah_{2,3}$ (B)	Re $Ah_{3,3}$ (B)	Im $Ah_{3,3}$ (B)
0	0.2461627	0	0.2461627	0	0.1230814	-0.2131832	0.1230814	0.2131832
0.2	.2284942	0.0175193	.2291648	0.0765231	.0989409	- .2195909	.1295532	.2371102
0.4	.2104389	.0345751	.2132603	.1628450	.0742057	- .2257614	.1362332	.2603364
0.6	.1917313	.0507078	.1983234	.2585534	.0483458	- .2313504	.143856	.2820583
0.8	.1722365	.0654713	.1842604	.3632557	.0209117	- .2358939	.1513248	.3013653
1.0	.1519465	.0784454	.1710012	.4765786	- .0084494	- .2388057	.1603959	.3172511
1.2	.1309726	.0892524	.1584923	.5981733	- .0399698	- .2393763	.1709423	.3286286
1.4	.1095344	.0975742	.1466919	.7277138	- .0737306	- .2367773	.1832650	.3343515
1.6	.0879445	.1031700	.1355665	.8648971	- .1096242	- .2300725	.1975686	.3332425
1.8	.0665884	.1058917	.1250882	1.0094440	- .1473116	- .2182389	.2139000	.3241306
2.0	.0459013	.1056958	.1152325	1.1610939	- .1861755	- .2002014	.2320768	.3058972
2.2	.0263413	.1026508	.1059766	1.3196055	- .2252729	- .1748338	.2516142	.2775346
2.4	.0083612	.0969395	.0972994	1.4847575	- .2632913	- .1412794	.2716525	.2382189
2.6	-.0076197	.0888538	.0891799	1.6563424	- .2985134	- .0985436	.2908937	.1873973
2.8	-.0212428	.0787835	.0815972	1.8341680	- .3287984	- .0461092	.3075556	.1248926
3.0	-.0322340	.0671991	.0745302	2.0180553	- .3515891	.0161746	.3193551	.0510245
3.2	-.0404226	.0546282	.0679575	2.2078401	- .3639534	.0878829	.3235308	-.0332548
3.4	-.0457537	.0416288	.0618576	2.4033645	- .3626736	.1678432	.3169198	-.1262143
3.6	-.0482937	.0287591	.0562083	2.6044851	- .3443934	.2539576	.2960996	-.2251985
3.8	-.0482275	.0165475	.0509874	2.8110662	- .3058345	.3430391	.2576069	-.3264915
4.0	-.0458482	.0054639	.0461727	3.0229778	- .2440898	.4306788	.1982415	-.4252148

TABLE 20

 $\beta = 142.5^\circ$

B	Re Ah _{1,3} (B)	Im Ah _{1,3} (B)	Ah _{1,3} (B)	Arg Ah _{1,3} (B)	Re Ah _{2,3} (B)	Im Ah _{2,3} (B)	Re Ah _{3,3} (B)	Im Ah _{3,3} (B)
0	0.2461627	0	0.2461627	0	0.1230814	-0.2131832	0.1230814	0.2131832
0.2	.2263826	0.0150510	.2268824	0.0663871	.1086419	- .2227151	.1264009	.2377662
0.4	.2064180	.0295347	.2085202	.1421172	.0762314	- .2322548	.1301866	.2617895
0.6	.1861864	.0429471	.1910754	.2267020	.0511817	- .2416840	.1350047	.2846311
0.8	.1657100	.0548587	.1745545	.3196962	.0241902	- .2507382	.1415197	.3055969
1.0	.1451043	.0649206	.1589652	.4206951	- .0053681	- .2589924	.1504724	.3239130
1.2	.1245630	.0728716	.1443128	.5293304	- .0380845	- .2658422	.1626475	.3387138
1.4	.1043394	.0785431	.1305976	.6452682	- .0744894	- .2704828	.1788288	.3490444
1.6	.0847266	.0818620	.1178134	.7682044	- .1150145	- .2718862	.1997411	.3537482
1.8	.0660352	.0828504	.1059473	.8978617	- .1599414	- .2687816	.2259766	.3516318
2.0	.0485723	.0816204	.0949798	1.0339877	- .2093340	- .2596398	.2579063	.3412602
2.2	.0326211	.0783668	.0848851	1.1763500	- .2629534	- .2426702	.2955745	.3210369
3.4	.0184227	.0733540	.0756320	1.3247372	- .3201536	- .2158322	.3385763	.2891862
2.6	.0061617	.0669013	.0671844	1.4789536	- .3797582	- .1768740	.3859200	.2437754
2.8	- .0040444	.0593653	.0595030	1.6388189	- .4399215	- .1234041	.4358770	.1827695
3.0	- .0121514	.0511206	.0525450	1.8041668	- .4979743	- .0530098	.4858228	.1041304
3.2	- .0181892	.0425408	.0462663	1.9748429	- .5502693	.0365660	.5320801	.0059748
3.4	- .0222584	.0339804	.0406215	2.1507038	- .5920342	.1471751	.5697758	- .1131947
3.6	- .0245226	.0257587	.0355650	2.3316159	- .6172548	.2799306	.5927322	- .2541719
3.8	- .0251972	.0181464	.0310515	2.5174542	- .6186163	.4347765	.5934190	- .4166301
4.0	- .0245358	.0113564	.0270365	2.7081022	- .5875384	.6099565	.5630027	- .5986001

TABLE 21

 $\beta = 150^\circ$

B	Re Ah _{1,3} (B)	Im Ah _{1,3} (B)	Ah _{1,3} (B)	Arg Ah _{1,3} (B)	Re Ah _{2,3} (B)	Im Ah _{2,3} (B)	Re Ah _{3,3} (B)	Im Ah _{3,3} (B)
0	0.2461627	0	0.2461627	0	0.1230814	-0.2131832	0.1230814	0.2131832
0.2	.2246140	0.0123381	.2249526	0.0548750	.1014306	- .2256758	.1231816	.2380139
0.4	.2031034	.0240868	.2045267	.1180424	.0791759	- .2384291	.1239275	.2625159
0.6	.1817259	.0347618	.1850208	.1890040	.0556420	- .2515939	.1260839	.2863557
0.8	.1606299	.0439923	.1665452	.2673188	.0300890	- .2651916	.1305409	.3091839
1.0	.1400053	.0515181	.1491831	.3525954	.0017008	- .2790958	.1383045	.3306140
1.2	.1200686	.0571856	.1329912	.4444875	- .0304148	- .2930052	.1504833	.3501908
1.4	.1010463	.0609394	.1179998	.5426840	- .0672239	- .3064064	.1682702	.3673458
1.6	.0831599	.0628134	.1042166	.6469053	- .1097563	- .3185263	.1929162	.3813396
1.8	.0666108	.0629186	.0916283	.7569011	- .1590848	- .3282723	.2256955	.3911909
2.0	.0515682	.0614292	.0802049	.8724449	- .2162875	- .3341608	.2678558	.3955900
2.2	.0381597	.0585673	.0699020	.9933301	- .2823899	- .3342346	.3205496	.3928018
2.4	.0264649	.0545874	.0606644	1.1193687	- .3582744	- .3259694	.3847394	.3805568
2.6	.0165127	.0497609	.0524292	1.2503896	- .4445533	- .3061756	.4610661	.3559365
2.8	.0082816	.0443609	.0451273	1.3862339	- .5413896	- .2708994	.5496712	.3152603
3.0	.0017032	.0386499	.0386874	1.5267574	- .6482566	- .2153347	.6499598	.2539846
3.2	- .0033320	.0328682	.0330367	1.6718257	- .7636202	- .1337616	.7602882	.1666298
3.4	- .0069669	.0272256	.0281029	1.8213141	- .8845318	- .0195346	.8775649	.0467602
3.6	- .0093688	.0218955	.0238157	1.9751087	-1.0061181	.1348483	.9967493	- .1129528
3.8	- .0107202	.0170118	.0201078	2.1330990	-1.1209594	.3375515	1.1102393	- .3205397
4.0	- .0112094	.0126679	.0169153	2.2951876	-1.2183561	.5970249	1.2071467	- .5843570

TABLE 22

 $\beta = 157.5^\circ$

B	Re Ah _{1,3} (B)	Im Ah _{1,3} (B)	Ah _{1,3} (B)	Arg Ah _{1,3} (B)	Re Ah _{2,3} (B)	Im Ah _{2,3} (B)	Re Ah _{3,3} (B)	Im Ah _{3,3} (B)
0	0.2461627	0	0.2461627	0	0.1230814	-0.2131832	0.1230814	0.2131832
0.2	.2232134	0.0094277	.2234124	0.0422112	.1032670	- .2284165	.1199464	.2378442
0.4	.2005149	.0183260	.2013506	.0911412	.0829854	- .2441317	.1175295	.2624577
0.6	.1783153	.0262839	.1802420	.1463475	.0616595	- .2607481	.1166558	.2870320
0.8	.1568546	.0330138	.1602912	.2074463	.0385831	- .2786115	.1182714	.3116253
1.0	.1363576	.0383412	.1416454	.2741036	.0129048	- .2979840	.1234528	.3363252
1.2	.1170261	.0421916	.1243995	.3460258	- .0163946	- .3190251	.1334207	.3612167
1.4	.0990318	.0445762	.1086017	.4229534	- .0505292	- .3417630	.1495610	.3863391
1.6	.0825104	.0455758	.0942609	.5046567	- .0909406	- .3660514	.1734510	.4116272
1.8	.0675573	.0453243	.0813528	.5909294	- .1393343	- .3915097	.2068915	.4368341
2.0	.0542261	.0439931	.0698274	.6815869	- .1977168	- .4174388	.2519429	.4614319
2.2	.0425289	.0417755	.0596146	.7764617	- .2684321	- .4427089	.3109610	.4844844
2.4	.0324383	.0388740	.0506304	.8754017	- .3541902	- .4656099	.3866284	.5044840
2.6	.0238917	.0354886	.0427815	.9782707	- .4580823	- .4836566	.4819739	.5191453
2.8	.0167967	.0318073	.0359698	1.0849410	- .5835690	- .4933370	.6003656	.5251443
3.0	.0110373	.0279990	.0300959	1.1952965	- .7344263	- .4897930	.7454636	.5177919
3.2	.0064807	.0242092	.0250616	1.3092312	- .9146257	- .4664200	.9211064	.4906292
3.4	.0029840	.0205569	.0207723	1.4266439	-1.1281146	- .4143741	1.1310986	.4349310
3.6	.0004002	.0171340	.0171387	1.5474447	-1.3784529	- .3219745	1.3788530	.3391086
3.8	- .0014159	.0140061	.0140775	1.6715468	-1.6682432	- .1739982	1.6668273	.1880043
4.0	- .0026030	.0112143	.0115124	1.7988704	-1.9982750	.0491317	1.9956720	- .0379174

TABLE 23

 $\beta = 165^\circ$

B	Re Ah _{1,3} (B)	Im Ah _{1,3} (B)	Ah _{1,3} (B)	Arg Ah _{1,3} (B)	Re Ah _{2,3} (B)	Im Ah _{2,3} (B)	Re Ah _{3,3} (B)	Im Ah _{3,3} (B)
0	0.2461627	0	0.2461627	0	0.1230814	- .2131832	0.1230814	0.2131832
0.2	.2221998	0.0063682	.2222910	0.0286521	.1054508	- .2308855	.1167490	.2372538
0.4	.1986623	.0123388	.1990451	.0620297	.0875724	- .2492222	.1110900	.2615610
0.6	.1759151	.0176153	.1767949	.0998026	.0690721	- .2688295	.1068430	.2864448
0.8	.1542573	.0220039	.1558188	.1416883	.0494280	- .2903491	.1048294	.3123530
1.0	.1339238	.0254013	.1363115	.1874436	.0279504	- .3144326	.1059734	.3398340
1.2	.1150875	.0277807	.1183930	.2368567	.0037519	- .3417455	.1113356	.3695262
1.4	.0978631	.0291761	.1021197	.2897420	-.0242951	- .3729682	.1221583	.4021444
1.6	.0823116	.0296675	.0874949	.3459357	-.0576168	- .4087935	.1399283	.4384610
1.8	.0684457	.0293663	.0744795	.4052920	-.0980133	- .4499145	.1664590	.4792808
2.0	.0562369	.0284025	.0630023	.4676802	-.1477580	- .4970026	.2039949	.5254051
2.2	.0456218	.0269137	.0529688	.5329829	-.2097231	- .5506656	.2553451	.5775794
2.4	.0365095	.0250364	.042692	.6010978	-.2875394	- .6113814	.3240489	.6364176
2.6	.0287887	.0228979	.0367845	.6719173	-.3857960	- .6793930	.4145847	.7022909
2.8	.0223342	.0206139	.0303933	.7453634	-.5102889	- .7545505	.5326232	.7751644
3.0	.0170134	.0182830	.0249745	.8213532	-.6683284	- .8360772	.6853418	.8543601
3.2	.0126911	.0159866	.0204116	.8998100	-.8691130	- .9222294	.8818041	.9382161
3.4	.0092344	.0137881	.0165948	.9806685	-1.1241808	-1.0098082	1.1334152	1.0235963
3.6	.0065165	.0117343	.0134223	1.0638632	-1.4479452	-1.0934643	1.4544617	1.1051986
3.8	.0044189	.0098565	.0108017	1.1493330	-1.8583173	-1.1647200	1.8627363	1.1745765
4.0	.0028338	.0081725	.0086499	1.2370236	-2.3774082	-1.2106013	2.3802419	1.2187737

TABLE 24

 $\beta = 172.5^\circ$

B	Re $Ah_{1,3}$ (B)	Im $Ah_{1,3}$ (B)	$Ah_{1,3}$ (B)	Arg $Ah_{1,3}$ (B)	Re $Ah_{2,3}$ (B)	Im $Ah_{2,3}$ (B)	Re $Ah_{3,3}$ (B)	Im $Ah_{3,3}$ (B)
0	0.2461627	0	0.2461627	0	0.1230814	-0.2131832	0.1230814	0.2131832
0.2	.2215864	0.0032091	.2216097	0.0144813	.1079415	- .2330374	.1136449	.2362465
0.4	.1975499	.0062053	.1976474	.0314010	.0928201	- .2535817	.1047298	.2597870
0.6	.1744907	.0088337	.1747141	.0505827	.0776284	- .2755717	.0968622	.2844054
0.8	.1527400	.0109975	.1531354	.0718772	.0621559	- .2998443	.0905841	.3108417
1.0	.1325322	.0126495	.1331345	.0951563	.0460595	- .3273358	.0864728	.3399853
1.2	.1140154	.0137837	.1148455	.1203095	.0288454	- .3591078	.0851699	.3728915
1.4	.0972632	.0144254	.0983271	.1472400	.0098408	- .3963817	.0874224	.4064770
1.6	.0822872	.0146223	.0835763	.1758625	- .0118489	- .4405833	.0941361	.4552056
1.8	.0690493	.0144361	.0705422	.2061016	- .0373990	- .4933992	.1064483	.5078354
2.0	.0574726	.0139360	.0591381	.2378898	- .0683494	- .5568464	.1258220	.5707824
2.2	.0474526	.0131925	.0492523	.2711667	- .1067184	- .6333582	.1541711	.6465507
2.4	.0388658	.0122733	.0407576	.3058777	- .1551590	- .7258902	.1940248	.7381636
2.6	.0315780	.0112404	.0335189	.3419725	- .2171689	- .8380496	.2487469	.8492901
2.8	.0254507	.0101478	.0273992	.3794070	- .2973771	- .9742516	.3228278	.9843995
3.0	.0203467	.0090409	.0222650	.4181389	- .4019291	-1.1399102	.4222759	1.1489512
3.2	.0161337	.0079559	.0179887	.4581304	- .5390096	-1.3416663	.5551433	1.3496222
3.4	.0126875	.0069204	.0144521	.4993443	- .7195485	-1.5876594	.7322360	1.5945799
3.6	.0098935	.0059541	.0115470	.5417548	- .9581788	-1.8878462	.9680724	1.8938003
3.8	.0076487	.0050695	.0091762	.5853208	-1.2745351	-2.2543670	1.2821838	2.2594366
4.0	.0058611	.0042736	.0072538	.6300211	-1.6950170	-2.7019546	1.7008781	2.7062282

TABLE 25

 $\beta = 180^\circ$

B	Re Ah _{1,3} (B)	Im Ah _{1,3} (B)	Ah _{1,3} (B)	Arg Ah _{1,3} (B)	Re Ah _{2,3} (B)	Im Ah _{2,3} (B)	Re Ah _{3,3} (B)	Im Ah _{3,3} (B)
0	0.2461627	0	0.2461627	0	0.1230814	-0.2131832	0.1230814	0.2131832
0.2	.2213811		.2213811		.1106906	- .2348341	.1106906	.2348341
0.4	.1971791		.1971791		.0985895	- .2571212	.0985895	.2571212
0.6	.1740186		.1740186		.0870092	- .2807904	.0870092	.2807904
0.8	.1522412		.1522412		.0761206	- .3067185	.0761206	.3067185
1.0	.1320798		.1320798		.0660399	- .3359333	.0660399	.3359333
1.2	.1136729		.1136729		.0568365	- .3696460	.0568365	.3696460
1.4	.0970794		.0970794		.0485396	- .4092973	.0485396	.4092973
1.6	.0822938		.0822938		.0411469	- .4566200	.0411469	.4566200
1.8	.0692608		.0692608		.0346304	- .5137243	.0346304	.5137243
2.0	.0578880		.0578880		.0289440	- .5832107	.0289440	.5832107
2.2	.0480576		.0480576		.0240287	- .6683182	.0240287	.6683182
2.4	.0396362		.0396362		.0198181	- .7731198	.0198181	.7731198
2.6	.0324832		.0324832		.0162415	- .9027790	.0162415	.9027790
2.8	.0264565		.0264565		.0132283	-1.0638894	.0132283	1.0638894
3.0	.0214182		.0214182		.0107091	-1.2649224	.0107091	1.2649224
3.2	.0172374		.0172374		.0086187	-1.5168213	.0086187	1.5168213
3.4	.0137930		.0137930		.0068965	-1.8337943	.0068965	1.8337943
3.6	.0109749		.0109749		.0054874	-2.2343725	.0054874	2.2343725
3.8	.0086846		.0086846		.0043422	-2.7428314	.0043422	2.7428314
4.0	.0068352		.0068352		.0034176	-3.3911041	.0034176	3.3911041

TABLE 26

k	Coefficients of B^k in the Series (I.2)
0	0.246162705
1	.124427392
3	- .136757058 $\times 10^{-1}$
4	- .345631644 $\times 10^{-2}$
6	.151952287 $\times 10^{-3}$
7	.274310829 $\times 10^{-4}$
9	- .703482810 $\times 10^{-6}$
10	- .101596603 $\times 10^{-6}$
12	.177647174 $\times 10^{-8}$
13	.217086759 $\times 10^{-9}$
15	- .281979642 $\times 10^{-11}$
16	- .301509388 $\times 10^{-12}$
18	.307167366 $\times 10^{-14}$
19	.293868800 $\times 10^{-15}$
21	- .243783624 $\times 10^{-17}$
22	- .212026551 $\times 10^{-18}$
24	.147212333 $\times 10^{-20}$
25	.117792528 $\times 10^{-21}$
27	- .699013928 $\times 10^{-24}$
28	- .519367407 $\times 10^{-25}$
30	.267821428 $\times 10^{-27}$
31	.186153193 $\times 10^{-28}$

TABLE 27

Zeros of $Ah_{1,3}(B)$ for B real	
k	B_k
1	3.3721344
2	5.8958433
3	7.9620250
4	9.7881268
5	11.4574233
6	13.0129143
7	14.4804371
8	15.8770394
9	17.2147133
10	18.5022978
11	19.7465442
12	20.9527565
13	22.1251975
14	23.2673582
15	24.3821434
16	25.4720025
17	26.5390256
18	27.5850140
19	28.6115343
20	29.6199596

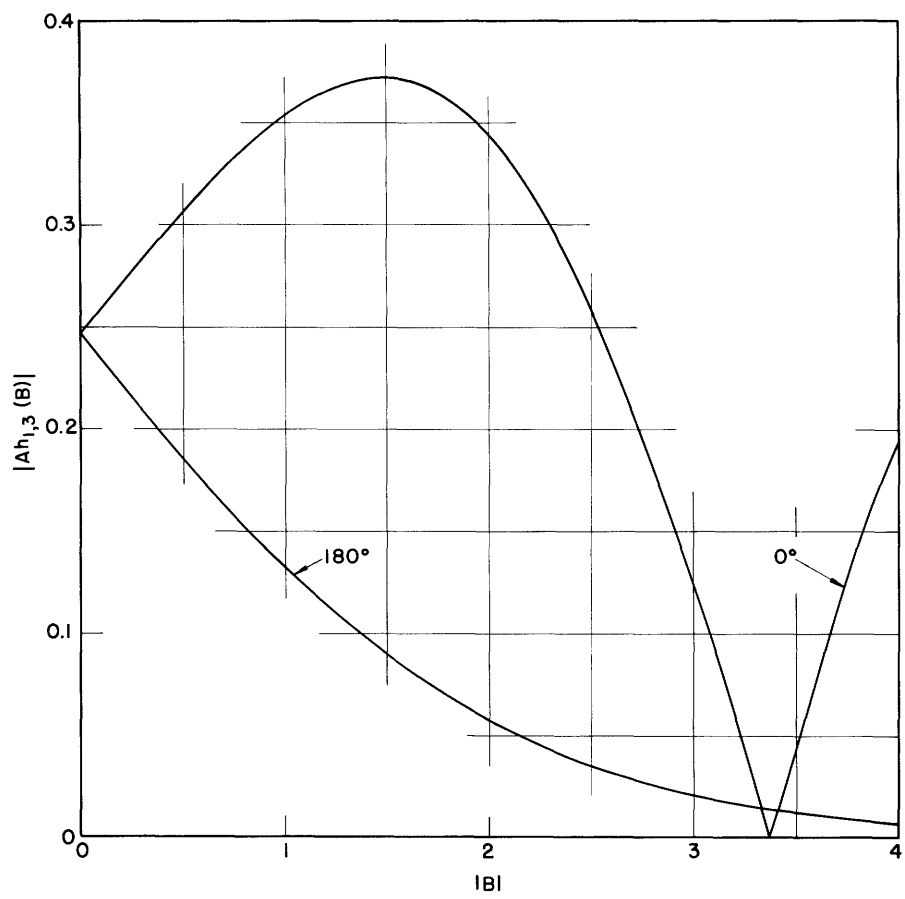


Fig. 3a

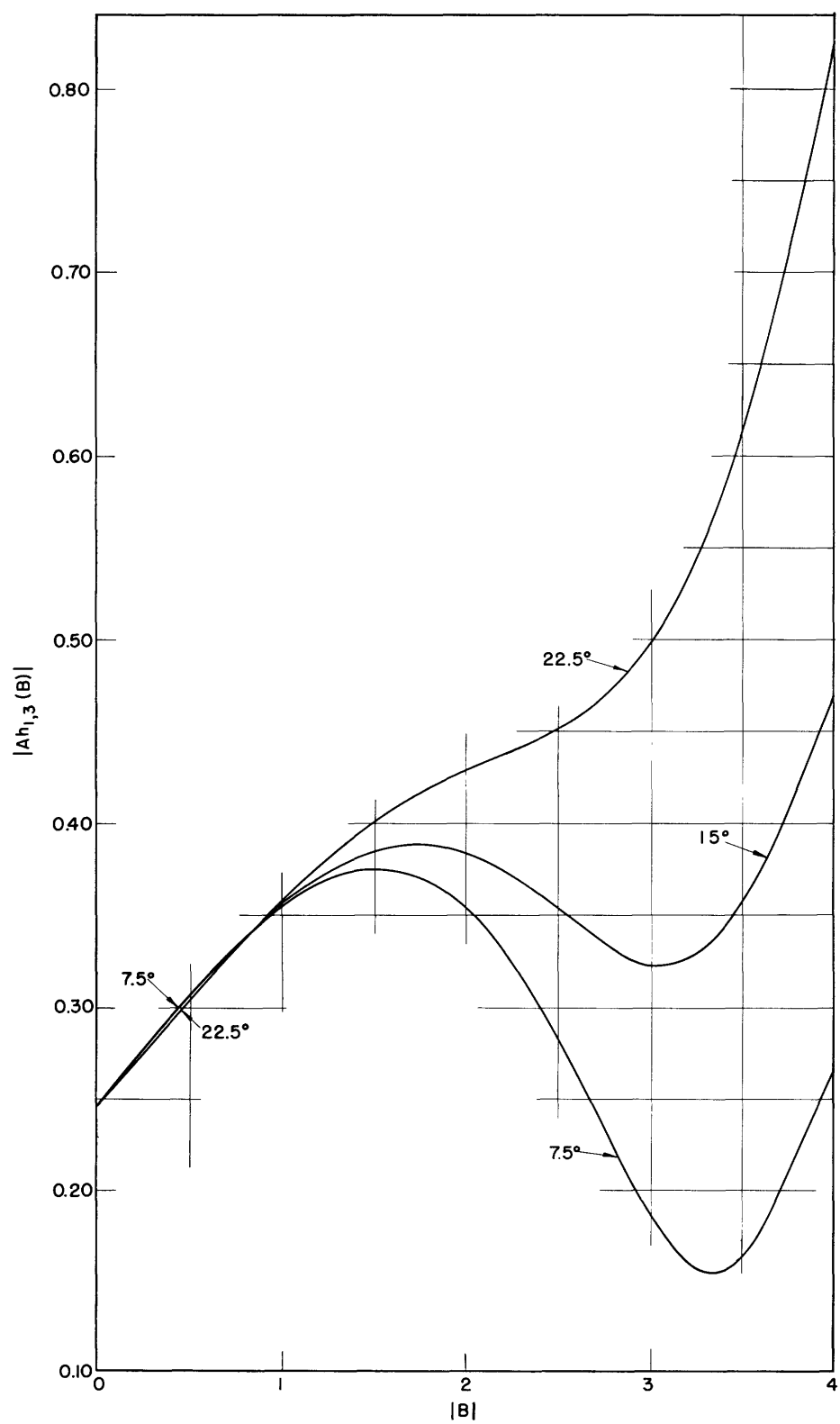


Fig. 3b

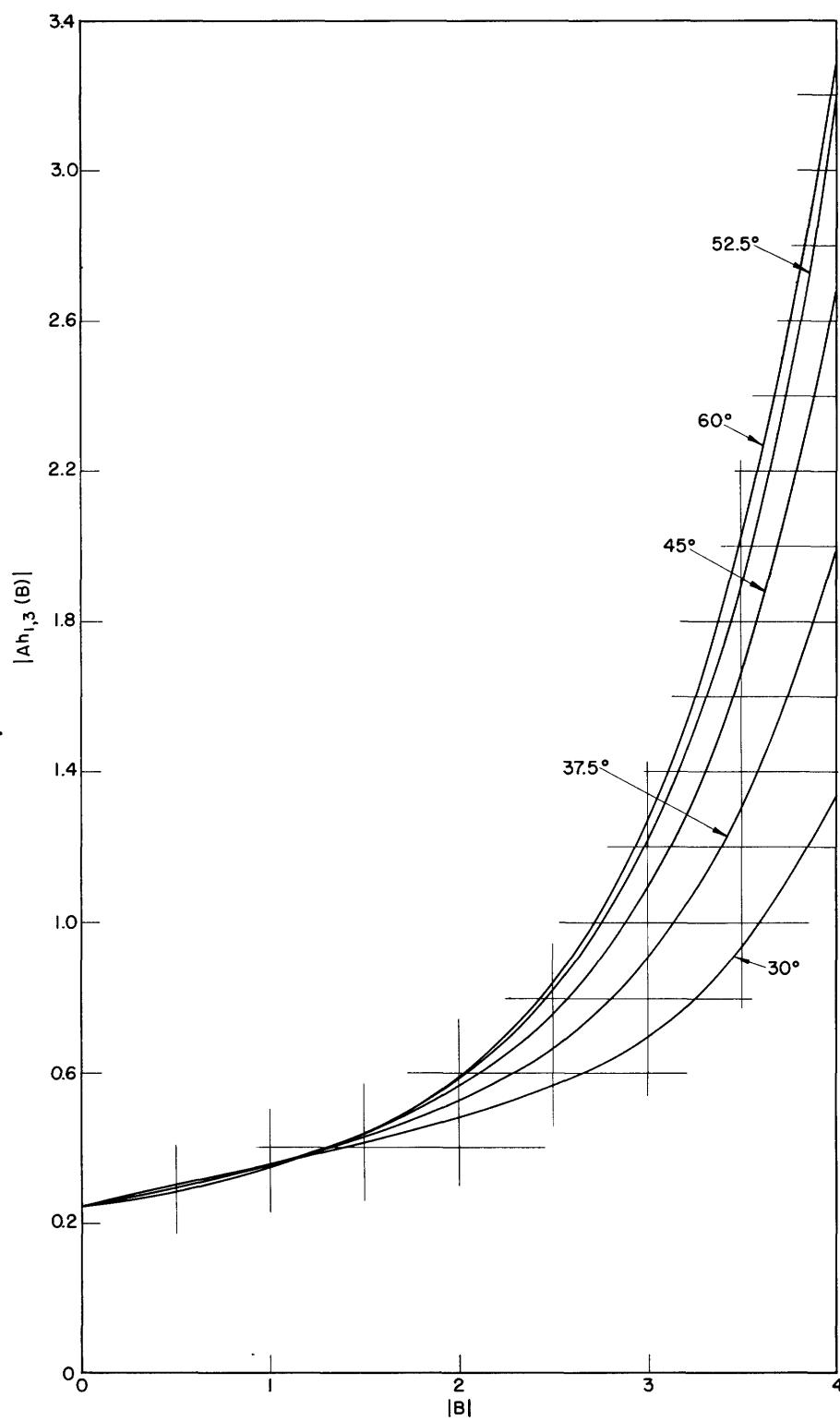


Fig. 3c

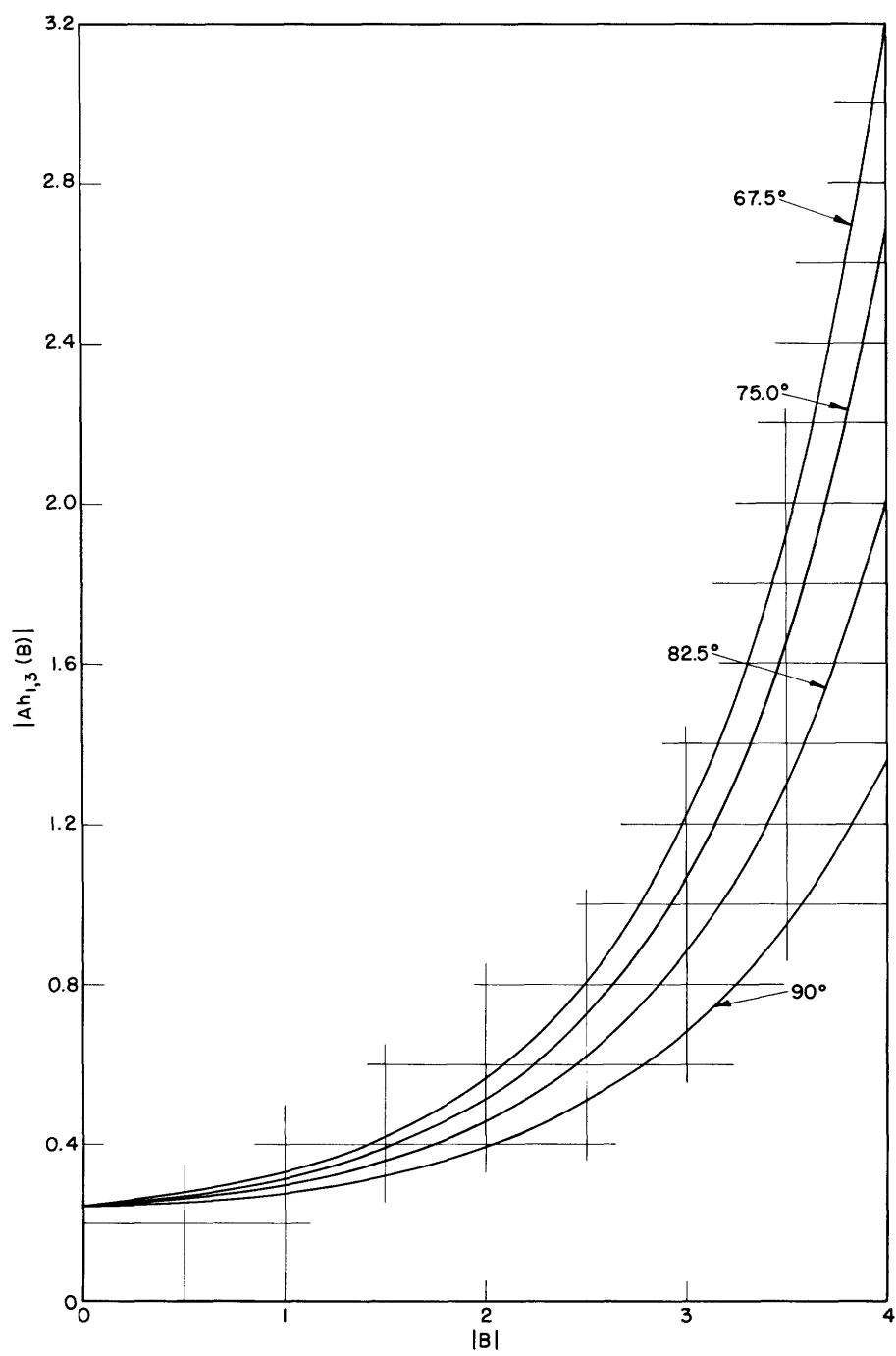


Fig. 3d

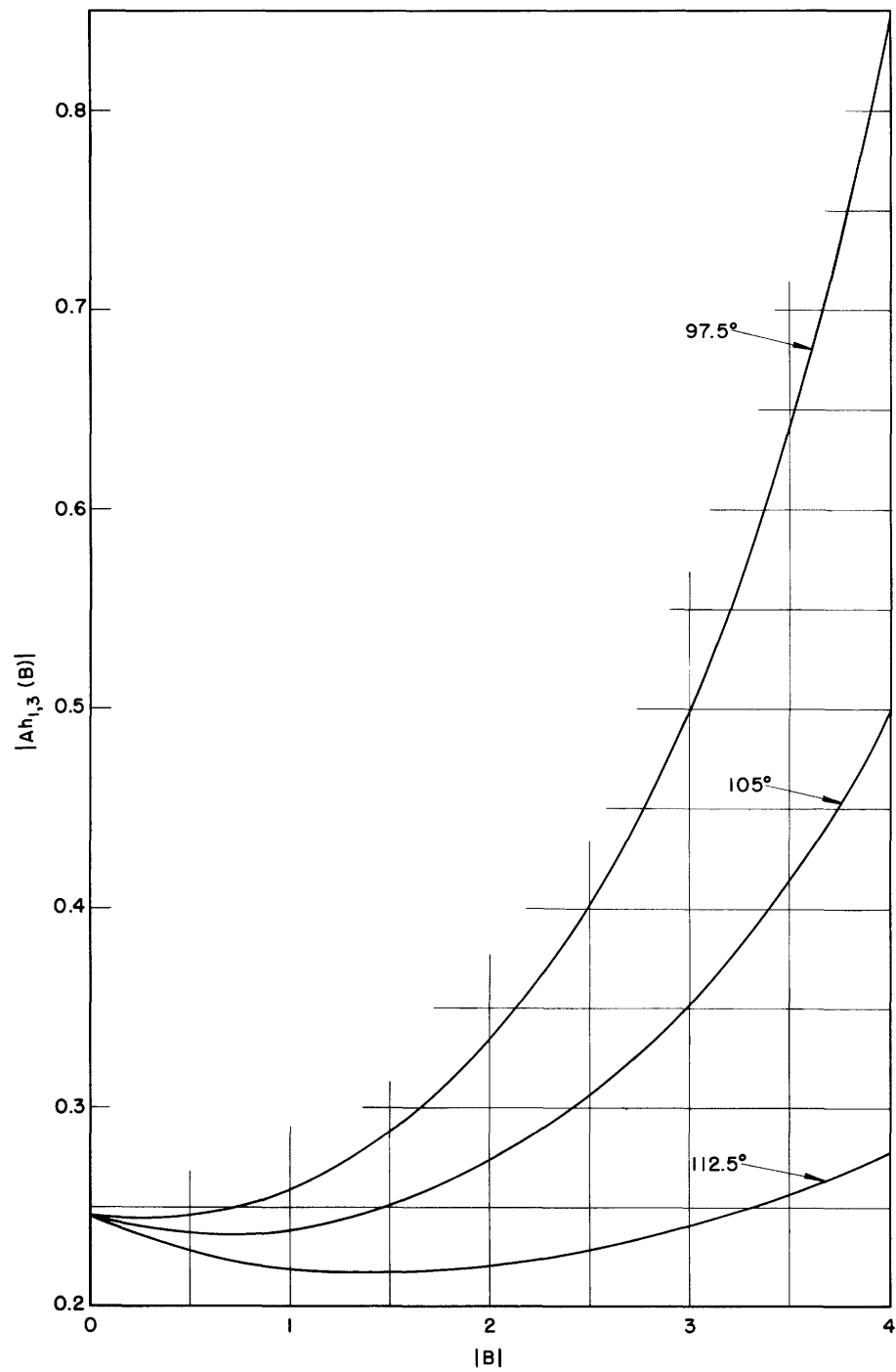


Fig. 3e

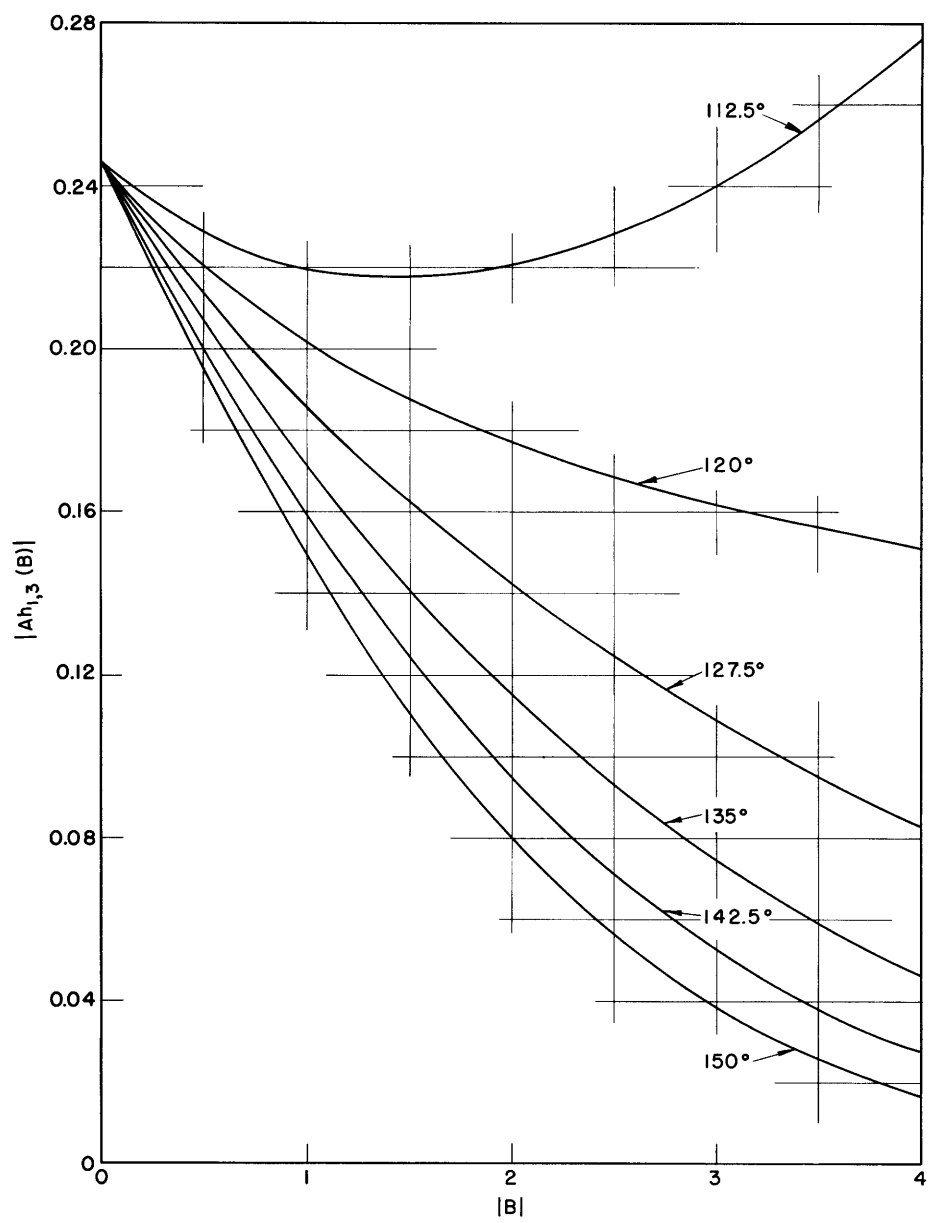


Fig. 3f

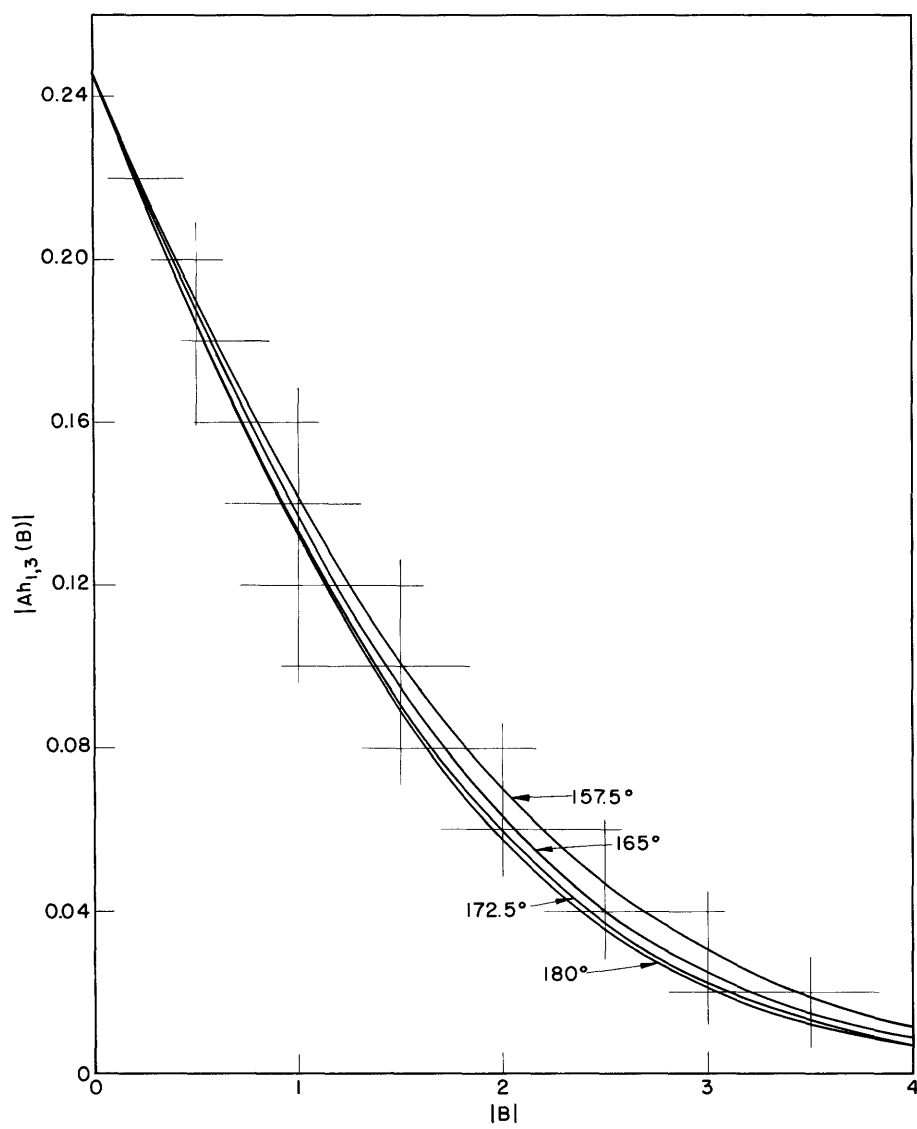


Fig. 3g

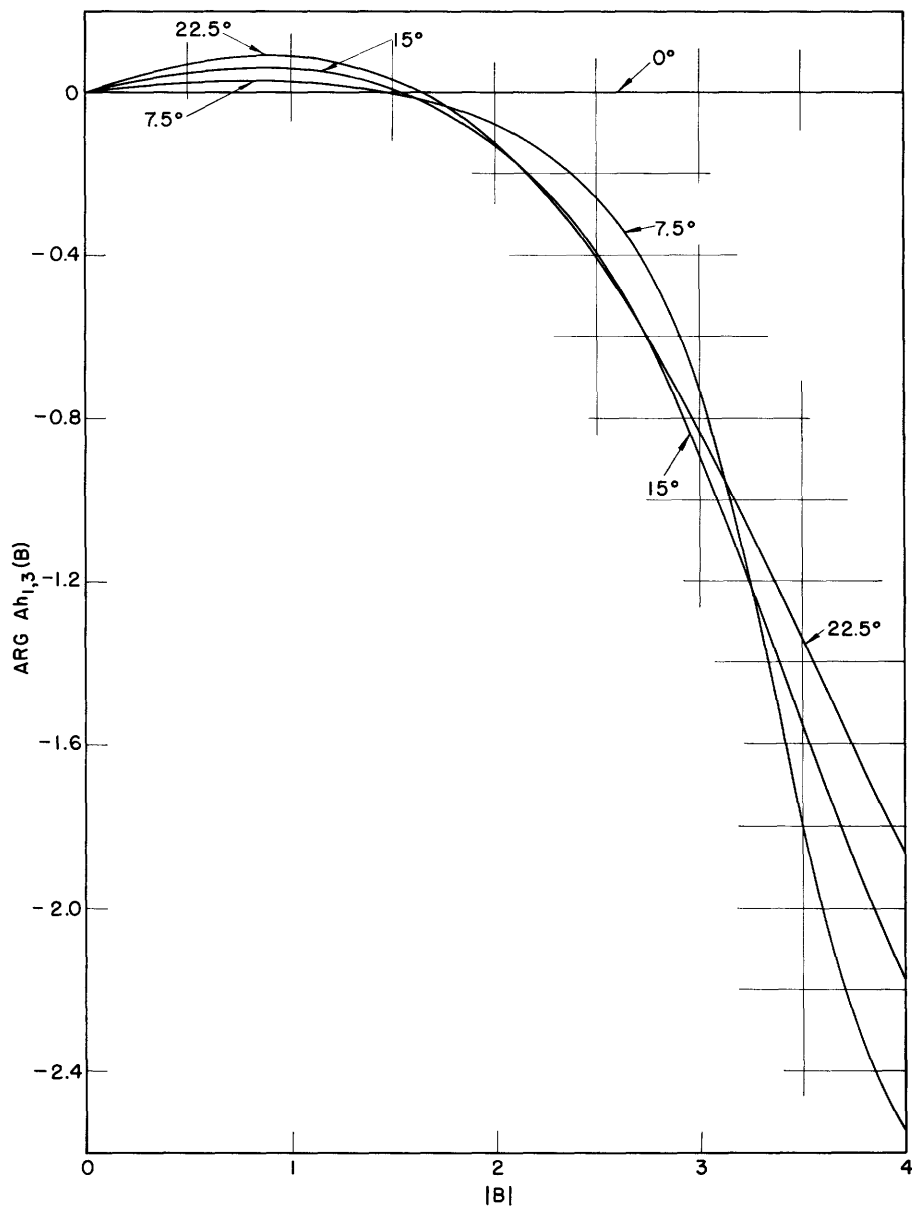


Fig. 4a

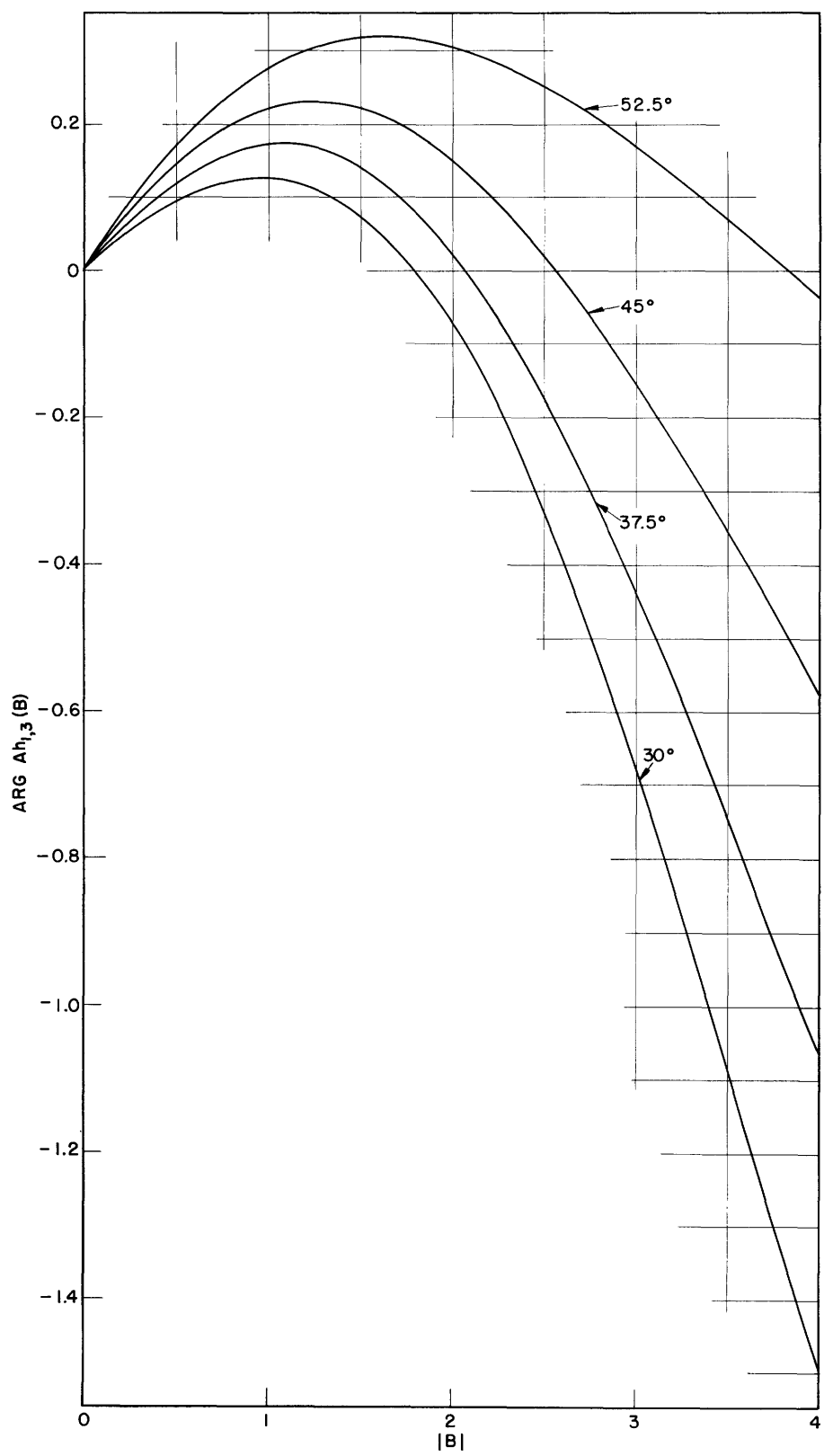


Fig. 4b

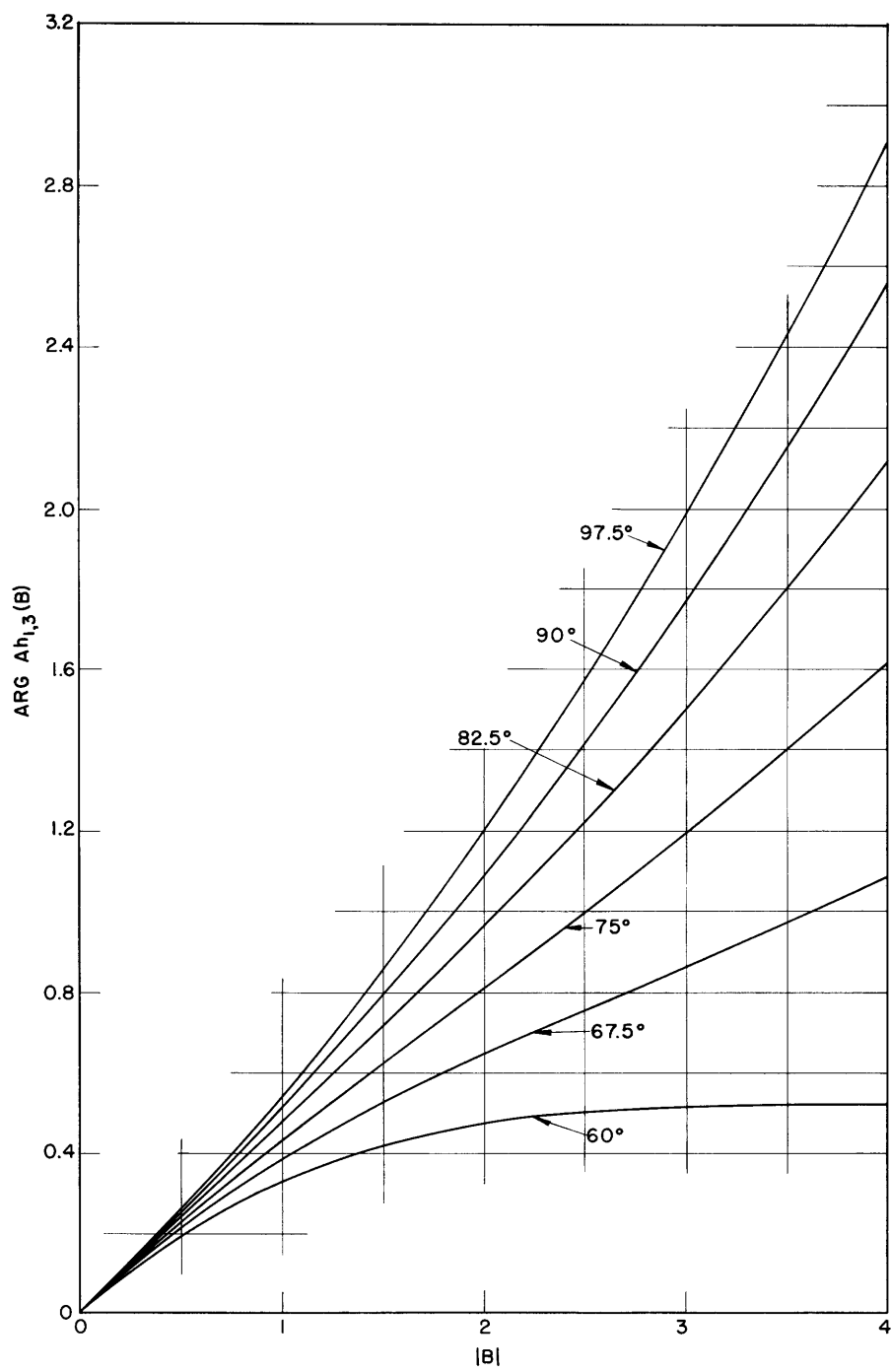


Fig. 4c

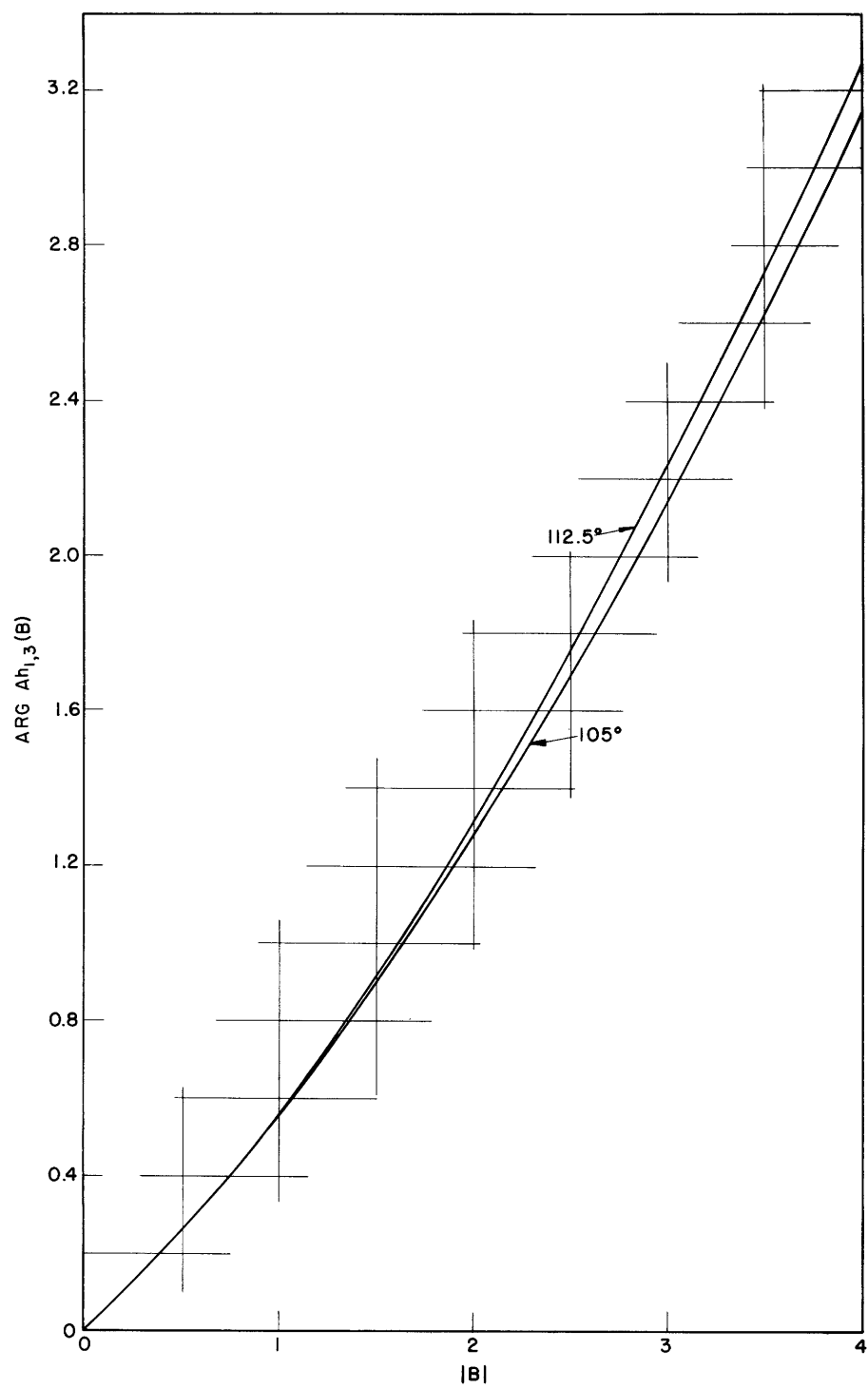


Fig. 4d

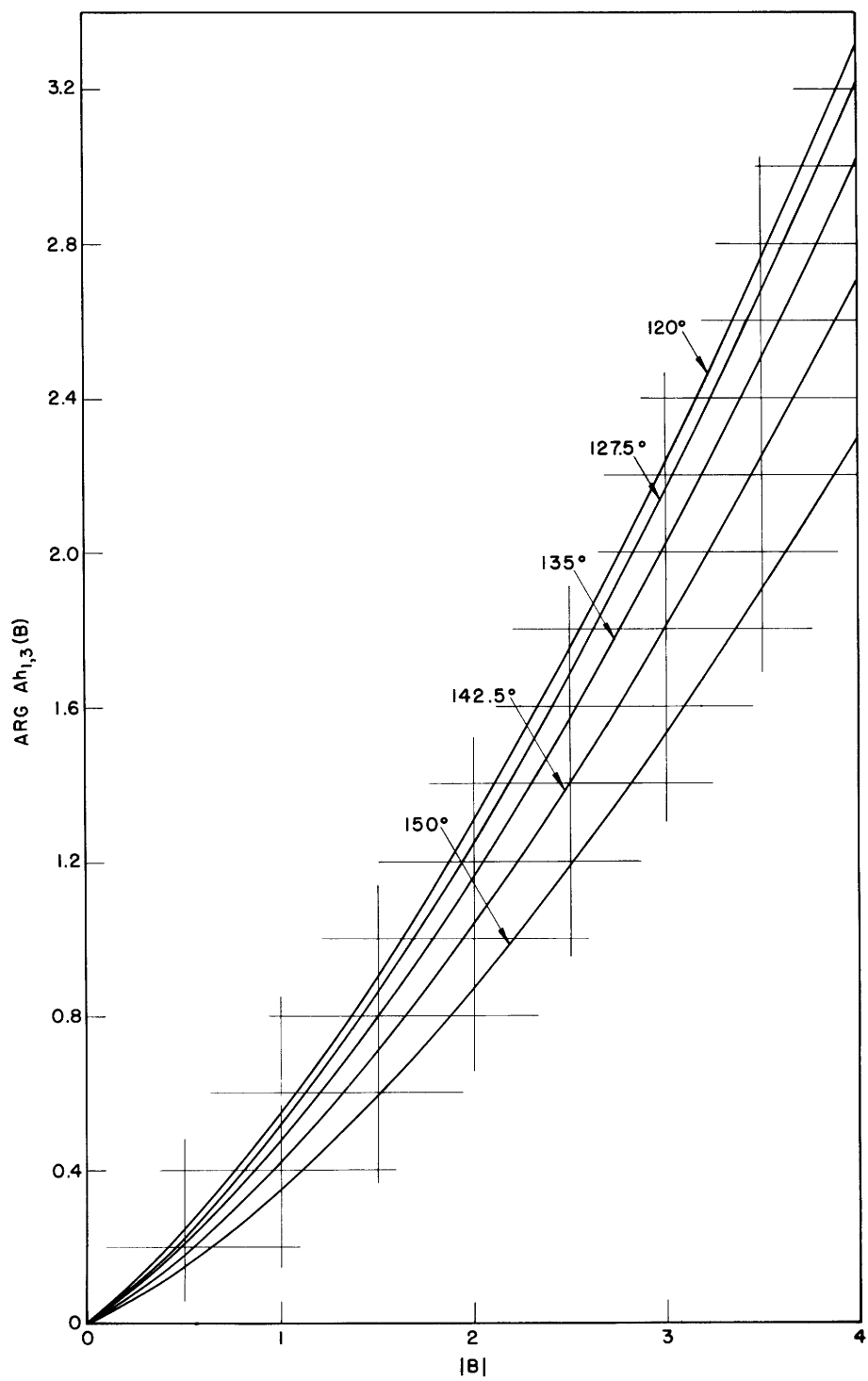


Fig. 4e

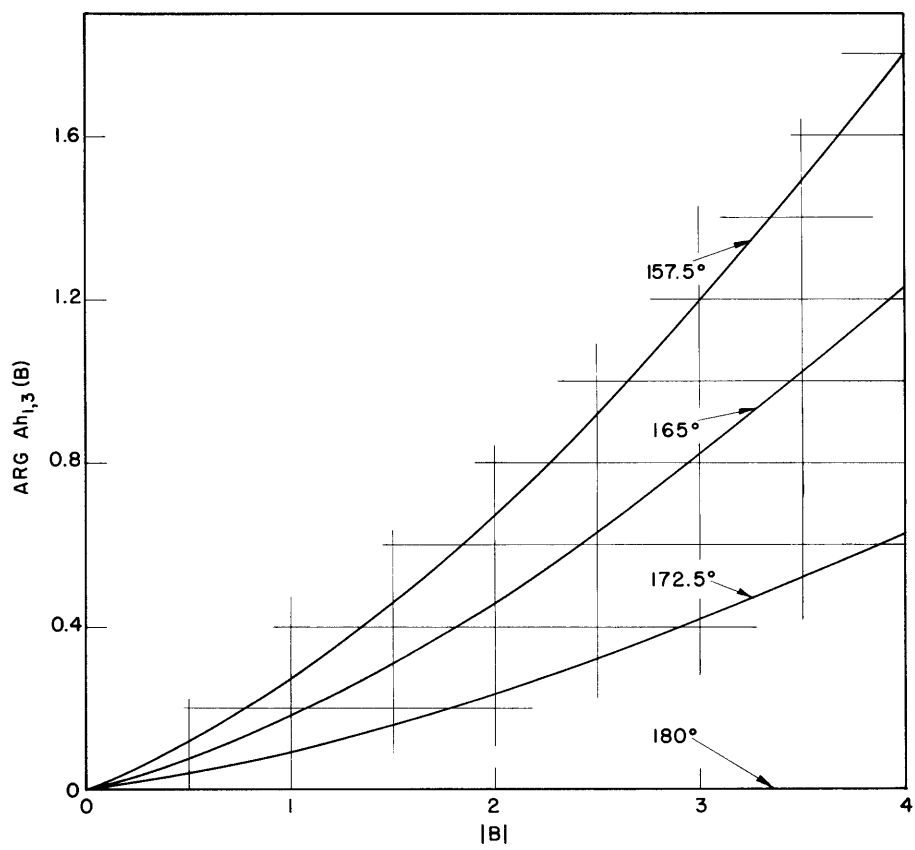


Fig. 4f

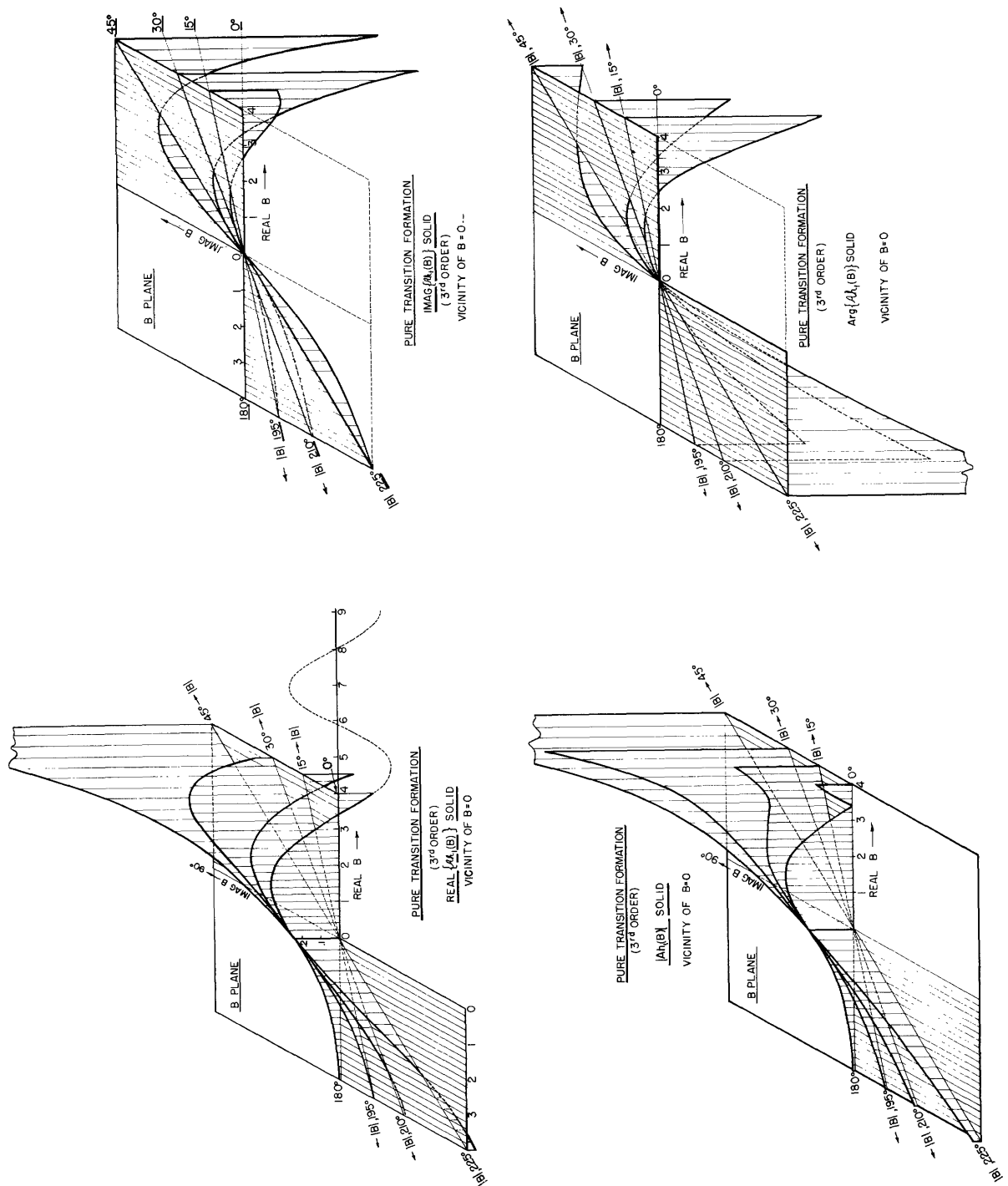


Fig. 5

APPENDIX I

APPLICATION OF THE EXTENDED AIRY-HARDY INTEGRALS TO THE EVALUATION, COMPOSITION, AND MECHANISM OF FORMATION OF TRANSIENT PHENOMENA OF THIRD ORDER

The material presented in Appendices I and II is intended to show an important application of the extended Airy-Hardy integrals to the evaluation and interpretation of transient phenomena in dispersive media when the controlling saddlepoints are of third order. A general discussion of this subject is found in reference 1; in fact, these two appendices are a reproduction of the material, with a few changes, of Chapter IV of reference 1.

IV.1 DEFINITIONS

IV.10 DEFINITIONS OF PURE TRANSITIONAL TRANSIENTS. Pure transitional transients are given, by definition, by

$$\frac{1}{2\pi i} \int_{\gamma} e^{W(s,t)} ds$$

We shall start with the discussion of mixed transients, in which the effect of $F(s)$ is considered. We shall first study simple typical cases which will help us in the attack of the general problem.

IV.11 THE COINCIDING POLE MIXED TRANSIENTS OF THE THIRD ORDER. Let $F(s)$ be reduced to

$$F(s) = \frac{1}{s - s_c} \tag{IV.1}$$

The denomination "coinciding" simply indicates that the pole is at $s = s_c$, which is the point of confluence.

The relation 1(I.3) allows us to treat only the case for the contour γ_1 which generates the $Ah_1(B)$ functions.

The corresponding integral for the coinciding pole transient of the third order is given by

$$f(t) = \frac{1}{2\pi i} \int e^{\frac{a+b(s-s_c)+c(s-s_c)^2+d(s-s_c)^3}{(s-s_c)}} ds = e^A \frac{1}{2\pi i} \int_{\gamma_1} \frac{e^{Bz + z^3}}{z} dz \tag{IV.1}$$

when the transformation $s - s_c = z/\sqrt[3]{d} - c/3d$ is introduced.

We shall study in some detail the canonical integral, which reads

$$\phi_{1,p}(B) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{e^{Bz + z^3}}{z} dz \quad 3(\text{IV.1})$$

In order to integrate 3(IV.1) one uses the auxiliary integral

$$\int_0^B e^{Bz} dB = \frac{e^{Bz}}{z} - \frac{1}{z} \quad 4(\text{IV.1})$$

Then 3(IV.1) becomes

$$\phi_{1,p}(B) = \frac{-1}{2\pi i} \int_{\gamma_1} \frac{e^{z^3}}{z} dz + \frac{1}{2\pi i} \int_{\gamma_1} e^{z^3} dz \int_0^B e^{Bz} dB = \frac{-1}{2\pi i} \int_{\gamma_1} \frac{e^{z^3}}{z} dz + \int_0^B Ah_{1,3}(B) dB \quad 5(\text{IV.1})$$

since

$$Ah_{1,3}(B) = \frac{1}{2\pi i} \int_{\gamma_1} e^{Bz + z^3} dz$$

The first integral of 5(IV.1) has a pole at $z = 0$. Figure 1(IV.1) shows the contour which is needed

$$\frac{1}{2\pi i} \int \frac{e^{z^3}}{z} dz = \frac{1}{2\pi i} \int_{\rho}^{\infty} \frac{e^{\tau^3}}{\tau} d\tau + \frac{1}{2\pi i} \int_{-\pi/3}^{+\pi/3} \frac{e^{\rho e^{i\phi}}}{\rho} d\phi - \frac{1}{2\pi i} \int_{\rho}^{\infty} \frac{e^{\tau^3}}{\tau} d\tau = \frac{1}{3} \quad 6(\text{IV.1})$$

Then, expression 5(IV.1) becomes, by using 6(IV.1) and 3(I.1),

$$\left. \begin{aligned} \phi_{1,p}(B) &= \frac{1}{3} + \sum_0^{\infty} \frac{\Gamma(\frac{\nu+1}{3})}{(\nu+1)!} B^{\nu+1} \sin \frac{\pi}{3} (\nu+1) \\ &= \frac{1}{3} \left\{ 1 + \frac{\sqrt{3}}{2\pi} \left[\Gamma\left(\frac{1}{3}\right) B + \frac{\Gamma(\frac{2}{3})}{2!} B^2 - \frac{\Gamma(\frac{4}{3})}{4!} B^4 + \dots \right] \right\} \end{aligned} \right\} \quad 7(\text{IV.1})$$

which is the corresponding solution of the coinciding pole transient.

Curve 2 in Fig. 2a(IV.1) shows the function $\phi_{1,p}(B)$ for B real. (In Figs. 2a, 2b, and 2c(IV.1) the point s_c was selected as the point s_d of confluence of the 2 saddlepoints which correspond to the third-order case.)

The complete pole transient can then be written as

$$f_p(t) = e^A \phi_{1,p}(B) \quad 8(\text{IV.1})$$

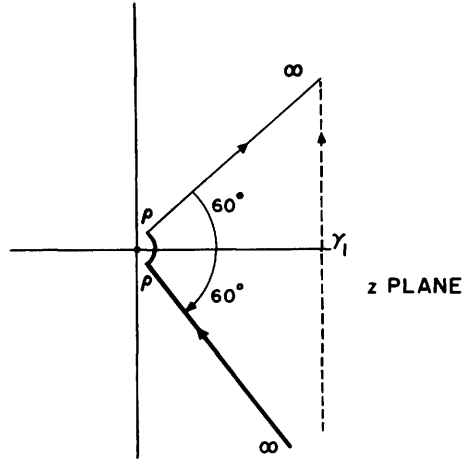


Fig. 1(IV.1)

IV.12 THE COINCIDING ZERO TRANSIENT. Let $F(s)$ be reduced to

$$F(s) = s - s_c \quad 9(\text{IV.1})$$

The coinciding zero transient is written as

$$f_o(t) = \frac{1}{2\pi i} \int (s-s_c) e^{a+b(s-s_c)+c(s-s_c)^2+d(s-s_c)^3} ds = \frac{e^A}{(\sqrt[3]{d})^2} \frac{1}{2\pi i} \int_{\gamma_1} z e^{Bz + z^3} dz \quad 10(\text{IV.1})$$

with canonical form

$$\phi_{1,0}(B) = \frac{1}{2\pi i} \int_{\gamma_1} z e^{Bz + z^3} dz \quad 11(\text{IV.1})$$

This integral can be computed immediately.

It can be seen that

$$\frac{1}{2\pi i} \int_{\gamma_1} z e^{Bz + z^3} dz = \frac{d}{dB} \frac{1}{2\pi i} \int_{\gamma_1} e^{Bz + z^3} dz = \frac{d}{dB} \text{Ah}_{1,3}(B)$$

since the function converges uniformly with respect to B .

Hence in accordance with 20(III.2) one gets

$$\left. \begin{aligned} \phi_{1,0}(B) &= \frac{1}{3\pi} \sum_{\nu=0}^{\infty} \frac{\Gamma\left(\frac{\nu+1}{3}\right)}{(\nu-1)!} B^{(\nu-1)} \sin \frac{\pi}{3} (\nu+1) \\ &= \frac{1}{2\pi\sqrt{3}} \left\{ \Gamma\left(\frac{2}{3}\right) - \frac{\Gamma\left(\frac{4}{3}\right)}{2!} B^2 - \frac{\Gamma\left(\frac{5}{3}\right)}{3!} B^3 + \dots \right\} \end{aligned} \right\} \quad 12(\text{IV.1})$$

Curve 3 in Fig. 2a(IV.1) shows the corresponding function 12(IV.1) for B real.

The complete zero transient is given by

$$f_o(t) = \frac{e^A}{(\sqrt[3]{d})^2} \phi_{1,0}(B) \quad 13(IV.1)$$

IV.13 COMPARISON OF PURE, COINCIDING POLE, AND COINCIDING ZERO TRANSIENT (third order). As a matter of illustration of the waveforms corresponding to the pure, coinciding pole, and coinciding zero transient, we offer Fig. 2a(IV.1) which is computed for only B real. Curve 1 shows the pure transitional transient.[†]

IV.14 SHIFTED POLE TRANSIENTS. (Small shift case). This is the case when

$$F = \frac{1}{s - s_b} \quad s_b \neq s_c \quad 14(IV.1)$$

Here we shall study the case of small shift, that is, s_b is close to s_c .

The corresponding integral reads

$$f_{s_b}(t) = \frac{1}{2\pi i} \int_{\gamma_s} \frac{e^{a + \dots + d(s-s_c)^3}}{(s-s_c) - (s_b-s_c)} ds = e^A \frac{1}{2\pi i} \int_{\gamma_1} \frac{e^{Bz + z^3}}{z - z_b} dz \quad 15(IV.1)$$

where $z_b = s_b - s_c$; z_b is small.

The corresponding canonical integral now is given by

$$\phi_{1,p_b} = \frac{1}{2\pi i} \int_{\gamma_1} \frac{e^{Bz + z^3}}{z - z_b} dz \quad 16(IV.1)$$

Now let us introduce the transformation

$$z - z_b = u \quad 17(IV.1)$$

We obtain

$$Bz + z^3 = u^3 + 3u^2 z_b + 3uz_b^2 + z_b^3 + Bu + Bz_b$$

or

$$\phi_{1,p_b}(B) = \frac{e^{z_b^3 + Bz_b}}{2\pi i} \int \frac{e^{u^3 + u(B + 3z_b^2)}}{u} e^{3u^2 z_b} du \quad 18(IV.1)$$

Let us introduce the new variable parameter

[†]In Fig. 2(IV.1) the variable v is used instead of z . This change in notation does not change the results.

$$X = B + 3z_b^2 \quad 19(\text{IV. 1})$$

(for small values of z_b , $X \approx B$).

Before performing the integration 18(IV. 1) it is convenient, for reasons of further simplification, to substitute this value in 15(IV. 1) so that we shall first consider the integral

$$\phi_{1,p_a}(X) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{e^{u^3+uX}}{u} e^{3u^2 z_b} du \quad 20(\text{IV. 1})$$

for small shifts.

$$e^{3u^2 z_b} \approx 1 + 3u^2 z_b \quad 21(\text{IV. 1})$$

so that 20(IV. 1) becomes

$$\phi_{1,p_a}(X) \approx \frac{1}{2\pi i} \int \frac{e^{uX+u^3}}{u} du + 3z_b \frac{1}{2\pi i} \int u e^{uX+u^2} du \quad 22(\text{IV. 1})$$

The first integral is a coinciding pole transient. The second is a coinciding zero transient. Then, the first theorem of composition is given by

"The shifted pole transient is equivalent to the sum of a coinciding pole transient and $3z_b$ times a coinciding zero transient." Since

$$X = B + 3z_b^2$$

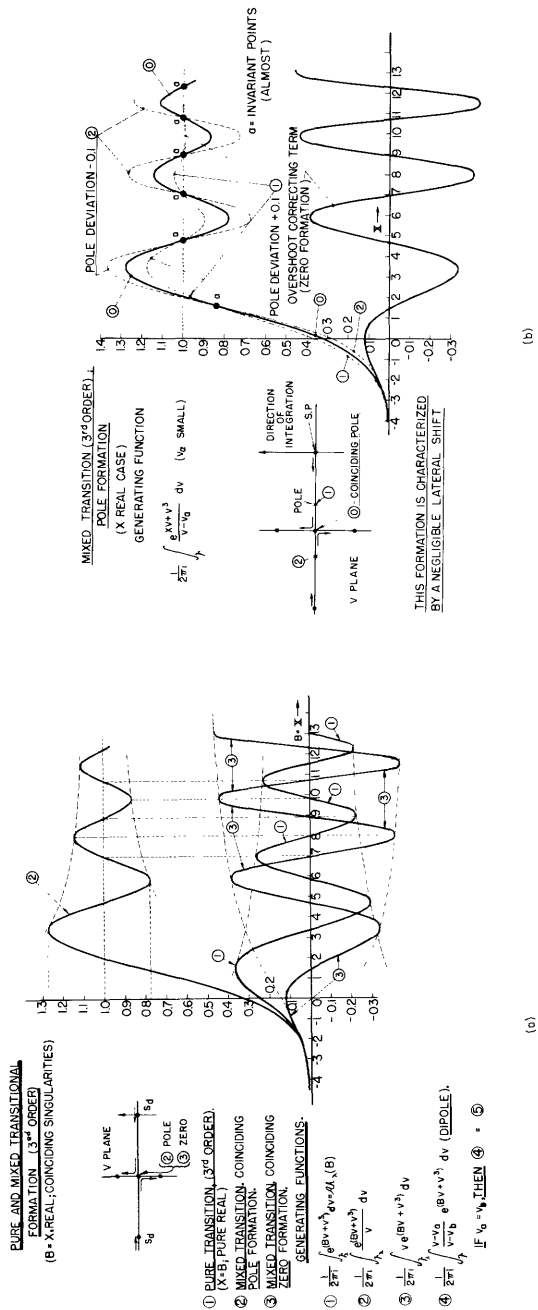
there is practically no shift if z_b is small. The function 22(IV. 1) is plotted in Fig. 2b(IV. 1). The curve 0 is the coinciding pole transient. The lower curve is the coinciding zero transient. Figure 2b(IV. 1) was computed for X real, $z_b = \pm 0.1$. Curve 1 is the final shifted pole transient for $z_b = -0.1$, curve 2 is the corresponding curve for $z_b = +0.1$. It can be noted immediately that

"In transitional transients, shown here for the third order, a small shift of the coinciding pole produces a strong effect on the wave envelope overshoot, and leaves the rising part of the wave practically unchanged."

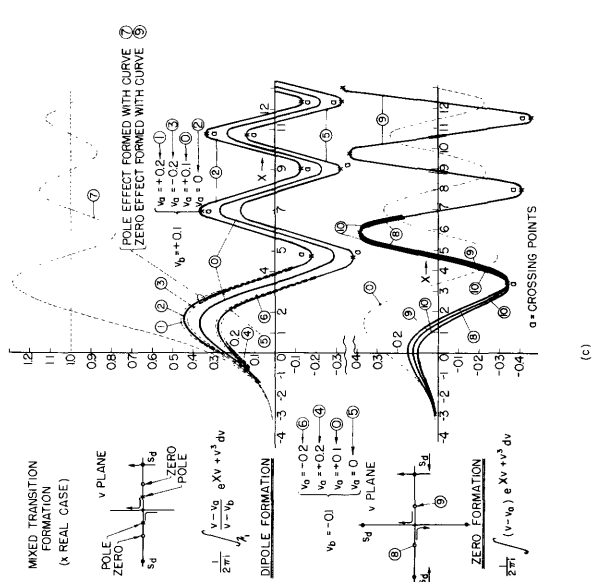
IV. 15 SHIFTED ZERO TRANSIENTS. This is the case when

$$F(s) = s - s_a \quad s_a \neq s_c \quad 23(\text{IV. 1})$$

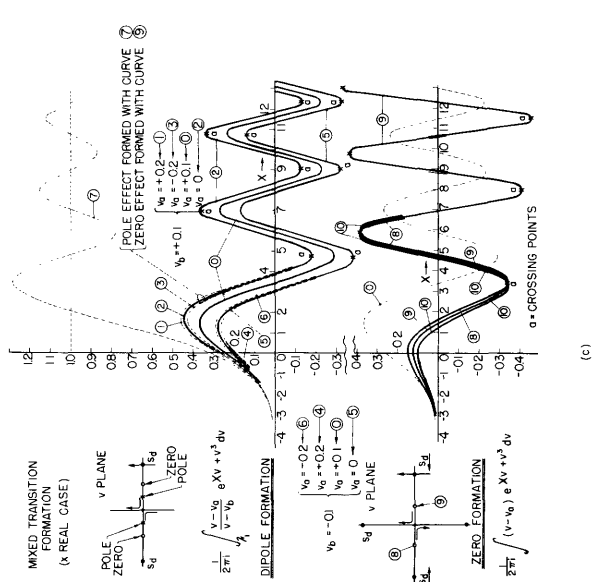
The shifted zero transient is given by



(a)



(b)



(c)

Fig. 2(IV.1)

$$\begin{aligned}
f_a(t) &= \frac{1}{2\pi i} \int (s - s_a) e^{a+(s-s_c)b+\dots+d(s-s_c)^3} ds \\
&= \frac{1}{2\pi i} \int_{\gamma} (s - s_c) e^{a+(s-s_c)b+\dots+d(s-s_c)^3} ds \\
&\quad - \frac{1}{2\pi i} (s_c - s_a) \int_{\gamma} e^{a+(s-s_c)b+\dots+d(s-s_c)^3} ds \\
&= \frac{e^A}{(\sqrt[3]{d})^2} \left\{ \frac{1}{2\pi i} \int_{\gamma} z e^{Bz + z^3} dz - \frac{1}{2\pi i} z_a \int_{\gamma} e^{Bz + z^3} dz \right\} \quad 24(IV. 1)
\end{aligned}$$

The first integral is a coinciding zero transient, the second is z_a times a pure transitional transient.

Figure 2c(IV. 1) shows the plot of

$$\frac{1}{2\pi i} \int z e^{Bz + z^3} dz - \frac{z_a}{2\pi i} \int e^{Bz + z^3} dz$$

for $z_a = \pm 0.01$ and B real.

In Fig. 2c(IV. 1), curve 9 is the coinciding zero transient. Curve 0 is the pure transitional transient. Curve 8 is the composite transient for a shift of $z_a = -0.01$ and curve 10 shows the effect of $z_a = 0.01$ zero displacement. It can be noted that

"In transitional transients (here proved for the third order) a small displacement of a zero from the coinciding position has a very small effect in the transient wave."

IV. 16 DIPOLE TRANSIENTS. This is the case where

$$\left. \begin{aligned} F(s) &= \frac{s - s_a}{s - s_b} \\ F(s) &= \frac{s - s_b + s_b - s_a}{s - s_b} = 1 + \frac{\delta}{s - s_b} \end{aligned} \right\} \quad \text{since} \quad 25(IV. 1)$$

so that

$$f_d(t) = \frac{1}{2\pi i} \int e^{a+\dots+d(s-s_c)^3} ds + \frac{(s_b - s_a)}{2\pi i} \int_{\gamma} \frac{e^{a+\dots+d(s-s_c)^3}}{s - s_b} ds \quad 26(IV. 1)$$

The first integral leads to

$$\frac{1}{2\pi i} \int_{\gamma_1} e^{a+\dots+d(s-s_c)^3} ds = \frac{e^A}{\sqrt[3]{d}} Ah_1(B) \quad 27(IV.1)$$

The second, in accordance with 20(IV.1), leads to

$$\begin{aligned} \frac{(s_b - s_a)}{2\pi i} \int e^{\frac{a+\dots+d(s-s_c)^3}{s-s_b}} ds = \delta e^{(A+Bz_b+z_b^3)} \left\{ \frac{1}{2\pi i} \int \frac{e^{uX+u^3}}{u} du \right. \\ \left. + 3z_b \frac{1}{2\pi i} \int u e^{uX+u^3} du \right\} \end{aligned} \quad 28(IV.1)$$

which is the composition of a coinciding pole and a coinciding zero transient.

The final dipole transient then reads

$$f_a(t) = \frac{e^A}{\sqrt[3]{d}} \left\{ Ah_1(B) + e^{Bz_b+z_b^3} [\delta \phi_{1,p}(X) + 3\delta z_b \phi_{1,0}(X)] \right\} \quad 29(IV.1)$$

when

$$X = B + 3z_b^2 \approx B \quad 30(IV.1)$$

The functions $\phi_{1,p}(X)$ are given by 7(IV.1) and 12(IV.1), respectively.

Figure 2c(IV.1) shows the dipole transient composition that corresponds to the values

$$z_a = \pm 0.2$$

$$z_b = \pm 0.1$$

from which the following cases can be separated.

Curve

$$\begin{aligned} \textcircled{1} \left\{ \begin{array}{ll} z_b = +0.1; & z_a = +0.2 \\ \textcircled{3} \left\{ \begin{array}{ll} z_b = +0.1; & z_a = -0.2 \\ \textcircled{4} \left\{ \begin{array}{ll} z_b = -0.1; & z_a = +0.2 \\ \textcircled{6} \left\{ \begin{array}{ll} z_b = -0.1; & z_a = -0.2 \end{array} \right. \end{array} \right. \end{array} \right. \end{array} \right. \end{aligned}$$

APPENDIX II

COMPARISON OF SECOND- AND THIRD-ORDER SOLUTIONS IN THE CASE OF COINCIDING POLES

It is of interest to compare integral solutions corresponding to the contribution of the second- and third-order saddlepoints in the case in which the saddlepoint orbit runs over a pole. Graphical comparison of the corresponding waveforms reveals at once the character of both types of transient formation.

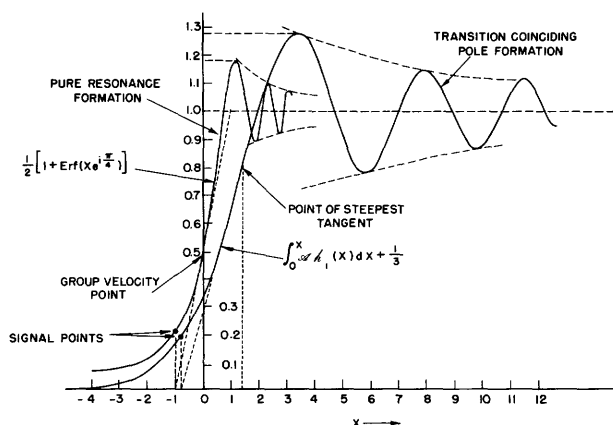


Fig. 1(IV.2)

Figure 1(IV.2) illustrates the graphs of the second- and third-order coinciding pole formation. Normalized units have been selected in order to give a proper basis of comparison.

It can be observed that both formations are characterized by the rapid increase of the response around the normalized point $X = 0$. For negative values of X , both cases show a slow monotonic increase reaching the signal point at practically the same value of X ($X = -1$). After this point, the second-order solution rises faster than the third-order solution. Both waves tend to the final state by oscillation. The second-order solution shows faster oscillation and smaller overshoot value, and converges faster than the wave corresponding to the third-order case.

The graphs of the third-order coinciding pole transients, illustrated in Fig. 1(IV-2), correspond to the functions associated with $Ah_{3,1}$. The cases associated with $Ah_{3,2}$ and $Ah_{3,3}$ are not shown.

